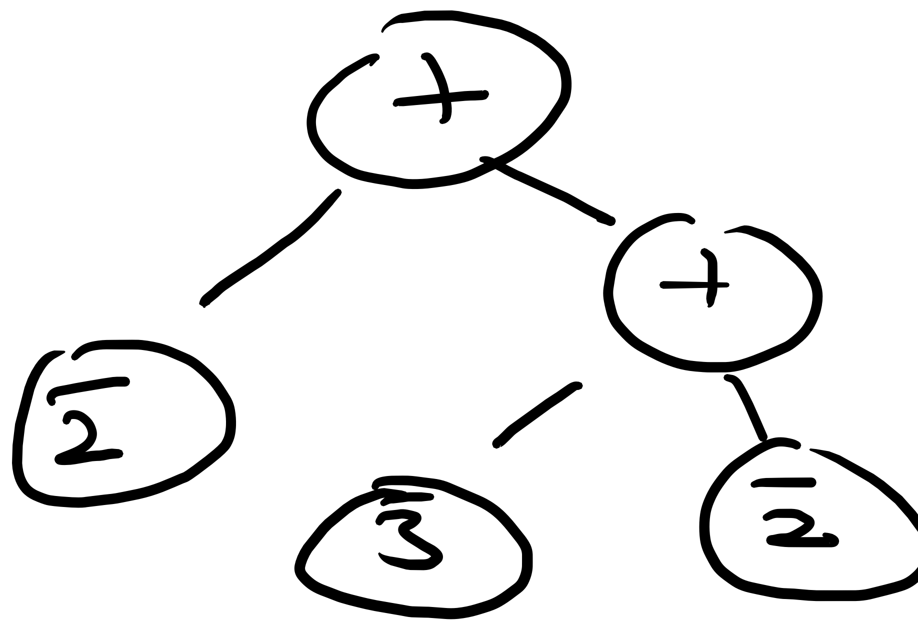
ADDITION Language

Syntax:

$$e ::= \overline{n} \mid e + e$$

$n \in \mathbb{N}$



Semantics Problem:

What is the meaning of an expression:

$$3+(2+4) \stackrel{?}{=} 9$$

Approach : Use the theory of ARS.

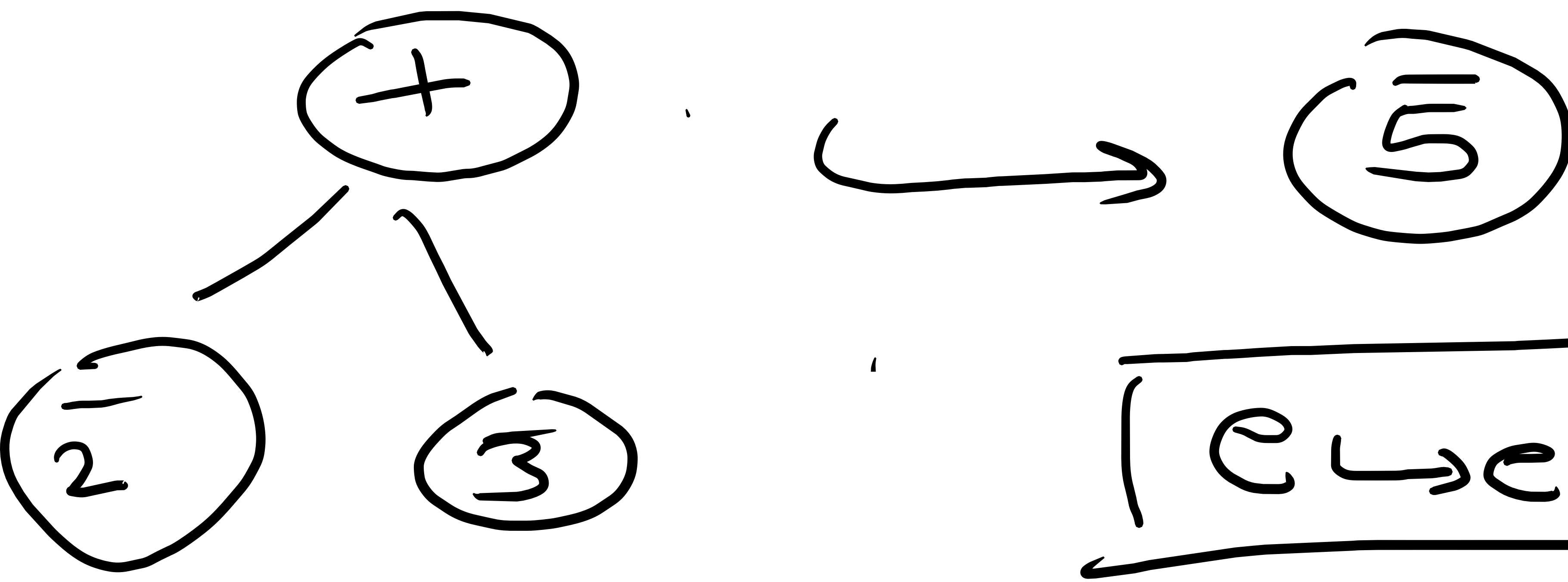
$$e \xrightarrow{*} \bar{n} \not\rightarrow$$

"Every expression simplifies to a unique number literal"

Is this true?

Rewrite Rules:

$$\underbrace{\bar{n}_1 + \bar{n}_2} \hookrightarrow \overline{n_1 + n_2} \qquad \text{PLUS}$$



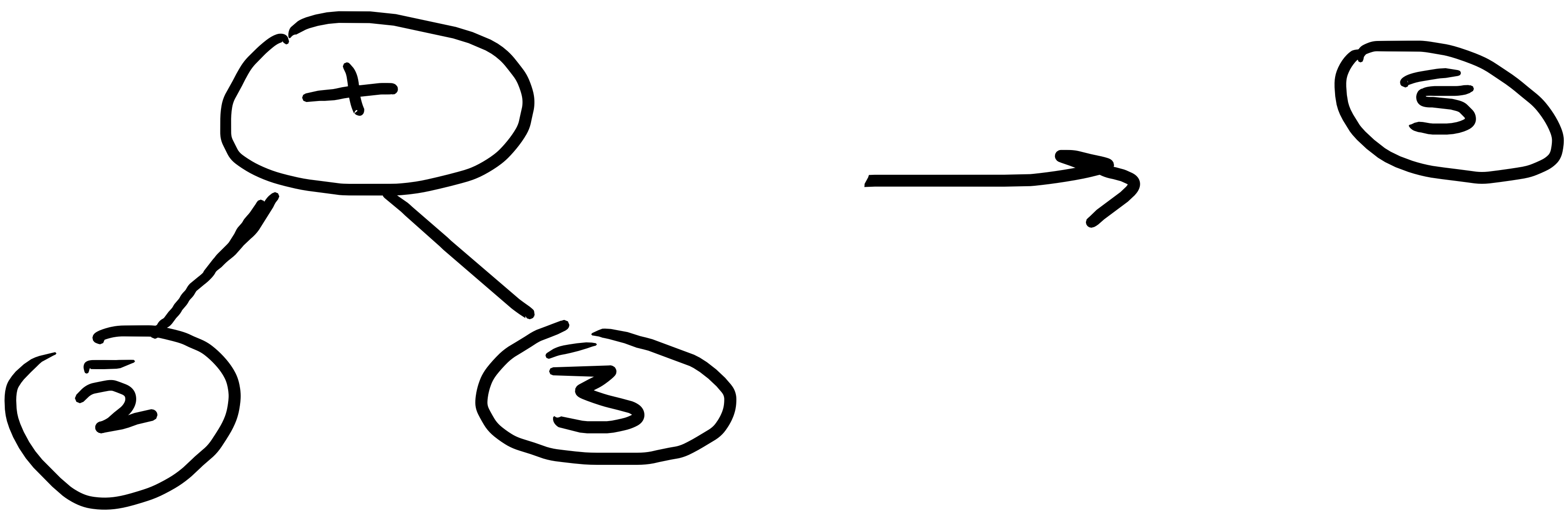
$$\boxed{e \hookrightarrow e'}$$

REDUCTION (SMALL STEP SEMANTICS)

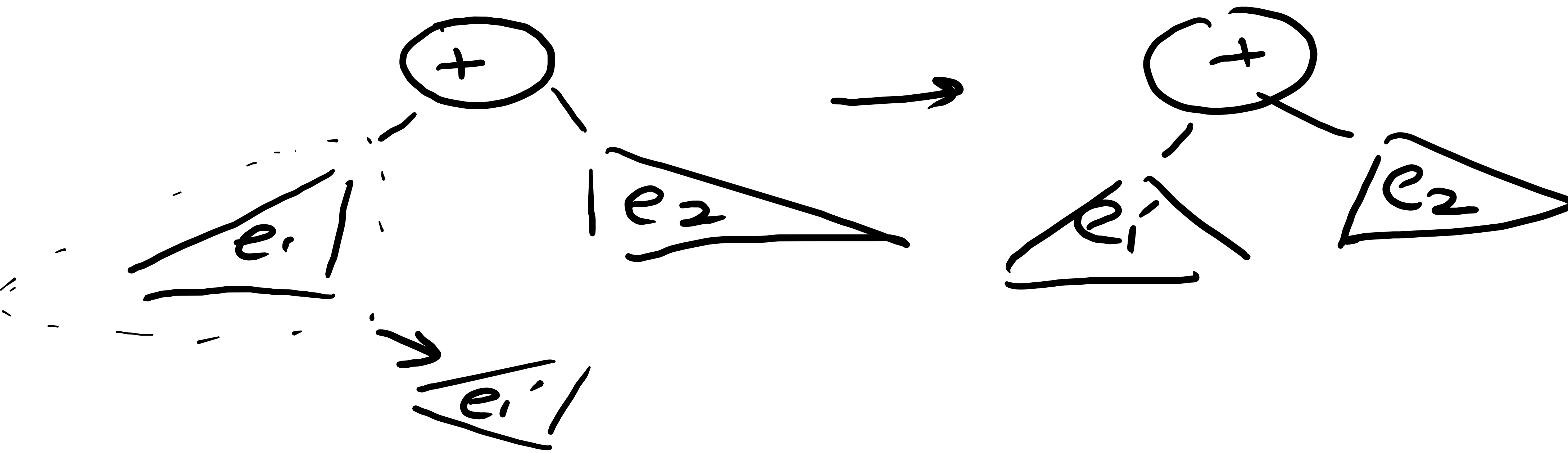
$$\boxed{e \longrightarrow e'}$$ Judgements

The relation $\longrightarrow$ is defined inductively:

$$\frac{e \hookrightarrow e'}{e \to e'}\ \text{REWRITE}$$



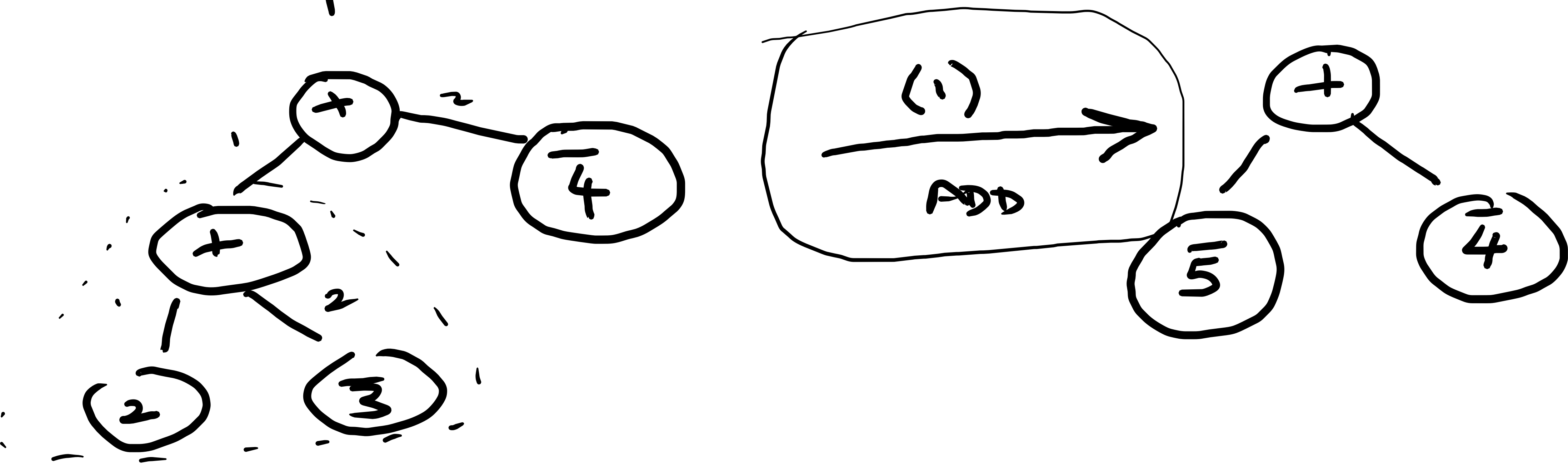
$$\frac{e_1 \longrightarrow e_1'}{e_1 + e_2 \longrightarrow e_1' + e_2}\ \text{LEFT}$$

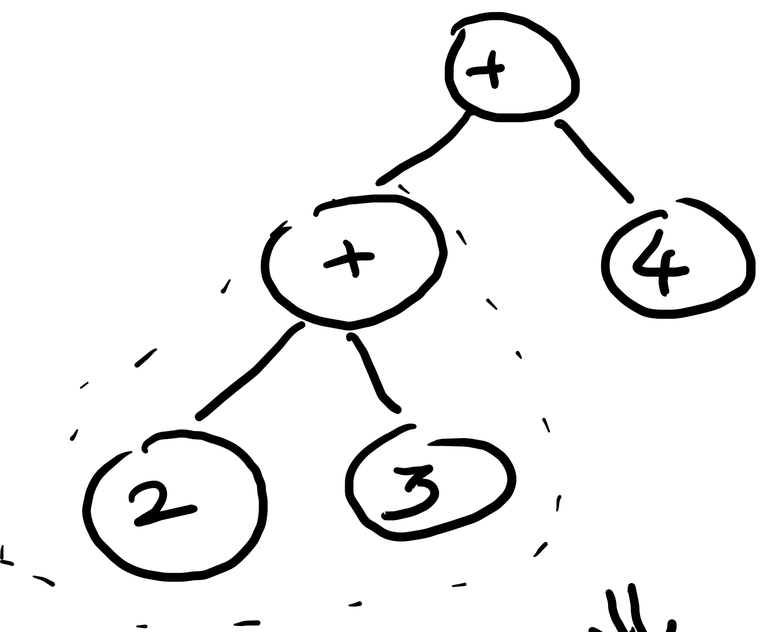


$$\frac{e_2 \longrightarrow e_2'}{e_1 + e_2 \longrightarrow e_1 + e_2'}$$

RIGHT

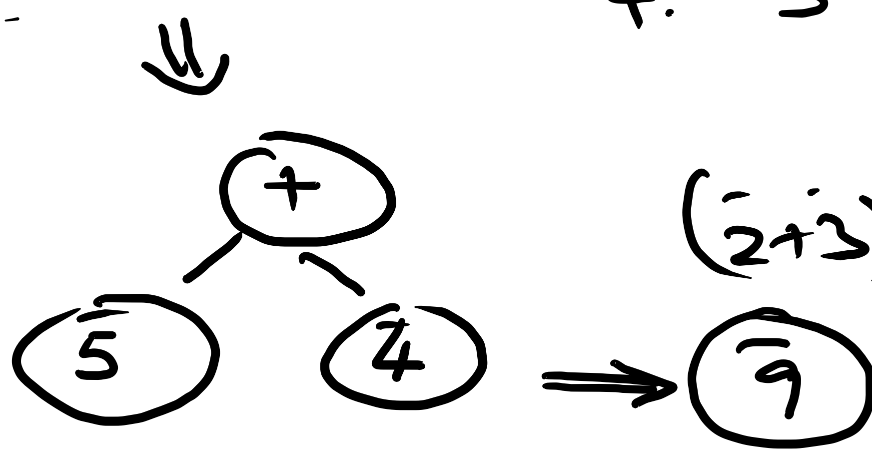
Example:





1. $\overline{2}+\overline{3} \hookrightarrow \overline{5}$ PLUS (rewrite)
2. $\overline{2}+\overline{3} \longrightarrow \overline{5}$ 1, REWRITE (REDUCE)
3. $(\overline{2}+\overline{3})+\overline{4} \longrightarrow \overline{5}+\overline{4}$ 2, LEFT (red)
4. $\overline{5}+\overline{4} \longrightarrow \overline{9}$

$(\overline{2}+\overline{3})+\overline{4} \xrightarrow{*} \overline{9}$ 3, 4, TRANSI



$$\boxed{e \to e'}$$

$$\frac{e_1 \hookrightarrow e_2}{e_1 \to e_2} \quad \text{REWRITE}$$

$$\frac{e_1 \to e_1'}{e_1 + e_2 \to e_1' + e_2} \quad \text{LEFT}$$

$$\frac{e_2 \to e_2'}{e_1 + e_2 \to e_1 + e_2'} \quad \text{RIGHT}$$

Simplification: The reflexive transitive closure of reduction

$$\frac{}{e \xrightarrow{*} e} \text{ REF}$$

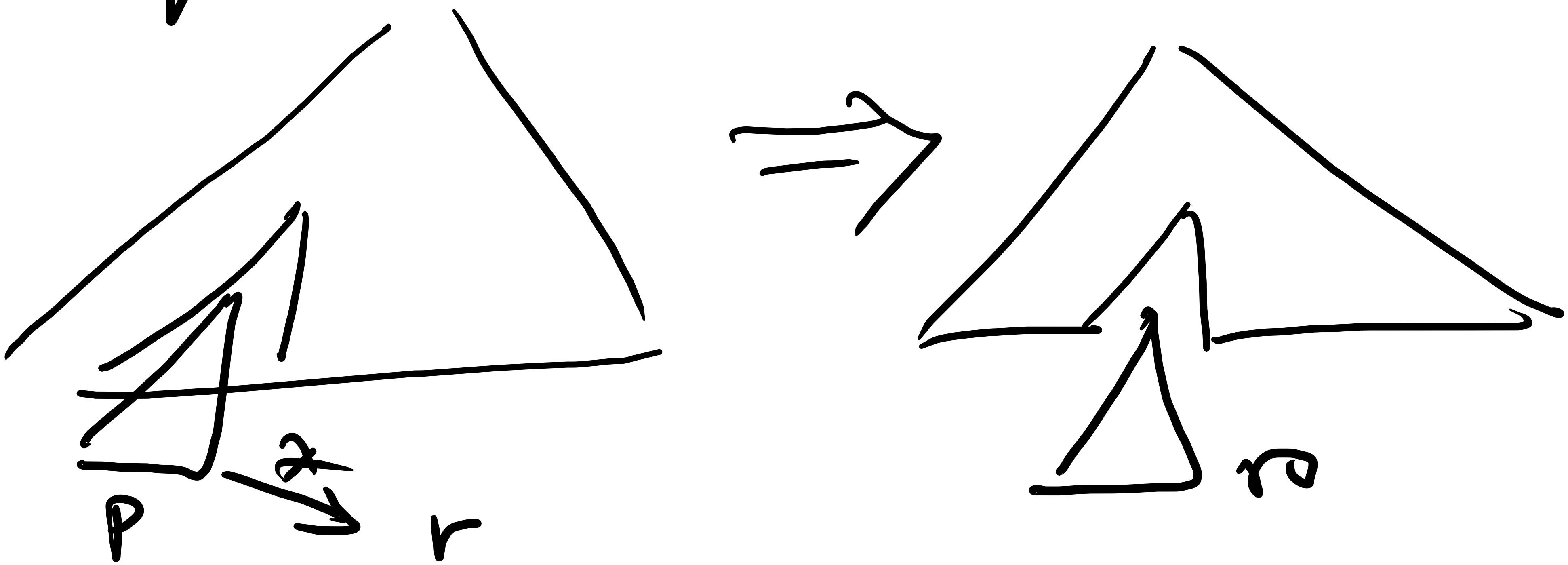
$$\frac{e_1 \to e_2 \quad e_2 \xrightarrow{*} e_3}{e_1 \xrightarrow{*} e_3} \text{ TRANS}$$

$$\frac{e \to e'}{e \xrightarrow{*} e'} \text{ RED}$$

Simplification Compatible Closure

$$p \xrightarrow{*} r \Rightarrow p + q \xrightarrow{*} r + q$$

$$q \xrightarrow{*} s \Rightarrow p + q \xrightarrow{*} p + s$$



Proof: By induction on dfn of $\xrightarrow{*}$

REF : $$\frac{p=r}{p \xrightarrow{*} r} \text{ REF}$$

1. $p=r$ given
2. $p+q=r+q$ 1, + is injective
3. $p+q \xrightarrow{*} r+q$ 2, REF

$$\cfrac{\cfrac{p=r}{p+q=r+q}\ \text{injectivity of } +}{p+q \xrightarrow{*} r+q}\ \text{REF}$$

ii) $$\frac{p \longrightarrow r}{p \xrightarrow{*} r}$$ RED.

RED, inversion

1. $p \longrightarrow r$

2. $p+q \longrightarrow r+q$ 1, LEFT

3. $p+q \xrightarrow{*} r+q$ 2, RED

(ii)

$$\frac{p \to s \qquad s \xrightarrow{*} r}{p \xrightarrow{*} r} \text{ TRANS}$$

1.	$s \xrightarrow{*} r$	TRANS, inv
2.	$s+q \xrightarrow{*} r+q$	1, Induction Hypothes
3.	$p \to s$	TRANS, inv
4.	$p+q \to s+q$	3, LEFT
5.	$p+q \xrightarrow{*} s+q$	4, RED
6.	$p+q \xrightarrow{*} r+q$	4, 2, TRANS

$$\frac{p \xrightarrow{*} r}{p + q \xrightarrow{*} r + q} \quad \text{LEFT}_*$$

$$\frac{p \xrightarrow{*} r}{s + p \xrightarrow{*} s + r} \quad \text{RIGHT}_*$$

VALUES:

$\bar{n}$: Numeric Literals.

$\bar{n} \not\rightarrow$ $\because$ there is no reduction rule in which $\bar{n}$ is the LHS.

$\bar{n}$ are normal form.

$e \not\rightarrow$ iff $e \equiv \bar{n}$

True?

Exercise Prove Thm.

'Contheerce,

Temmahm:

Termination $\longrightarrow$ is terminating :

$$e_0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow \cdots\cdots e_n \not\longrightarrow$$

Proof: by showing that

$$e \longrightarrow e' \Longrightarrow |e'| < |e|$$

(Ex: define $|e|$).

Confluence?:

$\mathbf{1}_{\mathcal{S}} \longrightarrow$ confluent?

(Exercise)