Abstract Reduction Systems

$$A = \langle X_A, \xrightarrow[A]{} \rangle$$

$$\xrightarrow[A]{} \subseteq X_A \times X_A$$

<u>Exercise</u>: Show that an ARS is a system

$$S = \langle X_S, X_S^0, U_S, \xrightarrow[S]{}, Y_S, h_S \rangle$$

$A_1 : \langle \mathbb{N}, \xrightarrow[A_1]{} \rangle$

$x \xrightarrow[A_1]{} y =_{df} y = x + 1$

$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$

$$A_2 = \langle \mathbb{N} \setminus \{0, 1\}, \xrightarrow[A_2]{} \rangle$$

$$x \xrightarrow[A_2]{} y =_{df} x = ky \text{ for some } k > 1$$

$$\begin{tikzcd}
& 24 \arrow[dl, dash] \arrow[dr] & & & \\
6 \arrow[dr] & & 8 \arrow[r] & 4 \arrow[r] & 2 \\
& 3 & & &
\end{tikzcd}$$

Relation Composition

$$\begin{tikzcd}
\text{LHS} \arrow[dr, squiggly] \arrow[rr, phantom, "\stackrel{?}{=}"] & & \text{RHS} \arrow[dl, squiggly] \\
& \text{ans} &
\end{tikzcd}$$

Relation Composition

$$R_{AB} \subseteq A \times B$$

$$R_{BC} \subseteq B \times C$$

$$R_{AB} ; R_{BC} =_{df} \{ (a,c) \in A \times C \mid \exists b \in B \ (a,b) \in R_{AB} \wedge (b,c) \in R_{BC} \}$$

$$\begin{tikzcd} a \arrow[r, "R_{AB}"'] \arrow[rr, bend right, "R_{AC}"'] & b \arrow[r, "R_{BC}"'] & c \end{tikzcd}$$

Derived AR systems

$A = (X_A, \to_A)$ $\quad x \xrightarrow[A]{} y$

$$A^0 = (X_A, \xrightarrow[A^0]{}) =_{df} \{(x, x) \mid x \in X_A\}$$

$$x \xrightarrow[A^0]{} y =_{df} x = y$$

$$x \xrightarrow{0} y$$

$$x \xrightarrow[A^{-1}]{} y \;=_{df}=\; y \xrightarrow[A]{} x$$

$$x \xrightarrow[A^{(n+1)}]{} y \;=_{df}=\; \xrightarrow[A]{} ; \xrightarrow[A^{(n)}]{}$$

$$\xrightarrow[A^{(0)}]{} \;:\; \{(x, x) \in A\}$$

Reflexive transitive closure

$$\xrightarrow{*} \; = \bigcup_{i \in \mathbb{N}} \xrightarrow[A]{(i)}$$

equivalently

$x \xrightarrow{*} y$ $=$df: if there is a path from x to y.

Transitive closure of A

$$x \xrightarrow{+} y \; =\text{df}: \bigcup_{i>0} \xrightarrow[A]{(i)}$$

$$\underset{A^{=}}{\longrightarrow} = \underset{A}{\longrightarrow} \cup \underset{A^{(0)}}{\longrightarrow}$$

$$\underset{A^{\leftrightarrow}}{\longrightarrow} \equiv \underset{A}{\longleftrightarrow} =_{df}= \underset{A}{\longrightarrow} \cup \underset{A^{-1}}{\longrightarrow}$$

$$\underset{A^{\leftrightarrow *}}{\longleftrightarrow} \equiv \underset{A}{\overset{*}{\longleftrightarrow}} =_{df}= \underset{(A^{\leftrightarrow})^{*}}{\longrightarrow}$$

$$\underset{A}{\overset{+}{\longleftrightarrow}} \quad \ldots$$

Exercise. for A_1 & A_2

define & interpret

A°

A^{i}

A^{*}

A^{+}

$A^{=}$

$A^{\leftrightarrow}$

$A^{\leftrightarrow *}$

$A^{\leftrightarrow +}$

Consider A_1: $x \xrightarrow[A_1]{} y \;=\!\text{df}\!=\; y = x+1$

$$x \xrightarrow[A_1^*]{} y \;=\!\text{df}\!=\; y \geqslant x$$

$A^{\leftrightarrow}$

$$x \underset{A_1}{\longleftrightarrow} y \;=\!\text{df}\!=\; \underbrace{x \xrightarrow[A_1]{} y}_{y=x+1} \text{ or } \underbrace{\overset{y \to x}{x \xrightarrow[A_1^{-1}]{} y}}_{x=y-1}$$

$$3 \xleftrightarrow{A_1} 4 \qquad \because \quad 3 \xrightarrow{A_1} 4$$

$$4 \xleftrightarrow{A_1} 3 \ ? \qquad 4 \xrightarrow{A_1^{-1}} 3$$

$$\therefore \quad 3 \xrightarrow{A_1} 4$$

$$n \longrightarrow n+1$$

$$n1 \longleftarrow n$$

$$0 \longrightarrow 1$$

$$\overset{*}{\underset{A_1}{\longleftrightarrow}} \equiv \underset{(A^{\leftrightarrow})^*}{\longrightarrow}$$

2. Interpret this derived relation

$$\text{LHS} \stackrel{?}{=} \text{RHS}$$

common

Terminology

① x reduces to y $=_{df.}$ $x \xrightarrow[A]{} y$

② x is reducible $=_{df.}$

$\exists y \in A : x \xrightarrow[A]{} y$.

$x \not\xrightarrow[A]{}$ x is not reducible
or irreducible or
x is a normal form

$$(3+4)+(5+1)$$
$$\downarrow$$
$$7+(5+1)$$
$$\downarrow$$
$$7+6$$
$$\downarrow$$
$$13 \not\longrightarrow$$

(3) y is a normal form of x if

1) $y \not\rightarrow$

2) $x \xrightarrow{*} y$

(4) x simblifies to y =df:

$x \xrightarrow{*} y$

⑥ y is a direct successor of x

if $x \xrightarrow[A]{} y$

⑦ y is a successor of x.

$\overset{\text{def}}{=}$ $x \xrightarrow[A]{+} y$

8 x and y are <u>joinable</u>

$x \downarrow y \quad =_{df}=$

$\exists z : \; x \xrightarrow{*} z$ and $y \xrightarrow{*} z$

$x \downarrow y$

$$\begin{tikzcd}
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
& z &
\end{tikzcd}$$

Exercise: For A_1 & A_2, identify & interpret normal forms, if any.

$$x \xrightarrow[A_1]{} y \quad \stackrel{\text{def}}{=} \quad y = x+1$$

What are the Normal forms of A_1?

$\infty \notin \mathbb{N}$ There are no normal forms.

A_1^{-1} : $x \xrightarrow[A_1^{-1}]{} y$ iff $y \xrightarrow[A_1]{} x$

i.e. $x = y+1$

What are the Normal Forms of A_1^{-1}?

$= \quad \{ 0$

$$\not\leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$$

$A_2 :$ $\quad x \xrightarrow[A_2]{} y \quad :\Leftrightarrow_{df} \quad x = k * y \quad k > 1$

$X_{A_2} := \mathbb{N} - \{0, 1\}$

Normal Forms of A_2?

<u>Primes</u>

Properties of AR systems

An ARS $A = (X_A, \xrightarrow[A]{})$ has the <u>Church</u> Rosser Property if

$\forall a,b \in X_A$

$$a \xleftrightarrow{*} b \Rightarrow a \downarrow b$$

$$\begin{tikzcd}
a \arrow[r, leftrightarrow, "*"] \arrow[dr, dashed, "*"'] & b \arrow[d, dashed, "*"] \\
& c
\end{tikzcd}$$

A1 : Is A1 Church Rosser?

$$n \overset{*}{\longleftrightarrow} m \Longrightarrow n \downarrow m$$

$m \downarrow m$ always.

$$\begin{tikzcd}
0 \arrow[r, leftrightarrow] & 1 \arrow[r, leftrightarrow] & 2 \arrow[r, leftrightarrow] & 3 \arrow[d, leftrightarrow] \\
 & & & 4
\end{tikzcd}$$

if $n \overset{*}{\longleftrightarrow} m$ then indeed $n \downarrow m$

Examples of an ARS that violates Church Rosser

$A = (X_A, \underset{A}{\rightarrow})$

$$X_A = \{a, b, c\}$$

$$\underset{A}{\rightarrow} = \{a \underset{A}{\rightarrow} b,\ a \rightarrow c\}$$

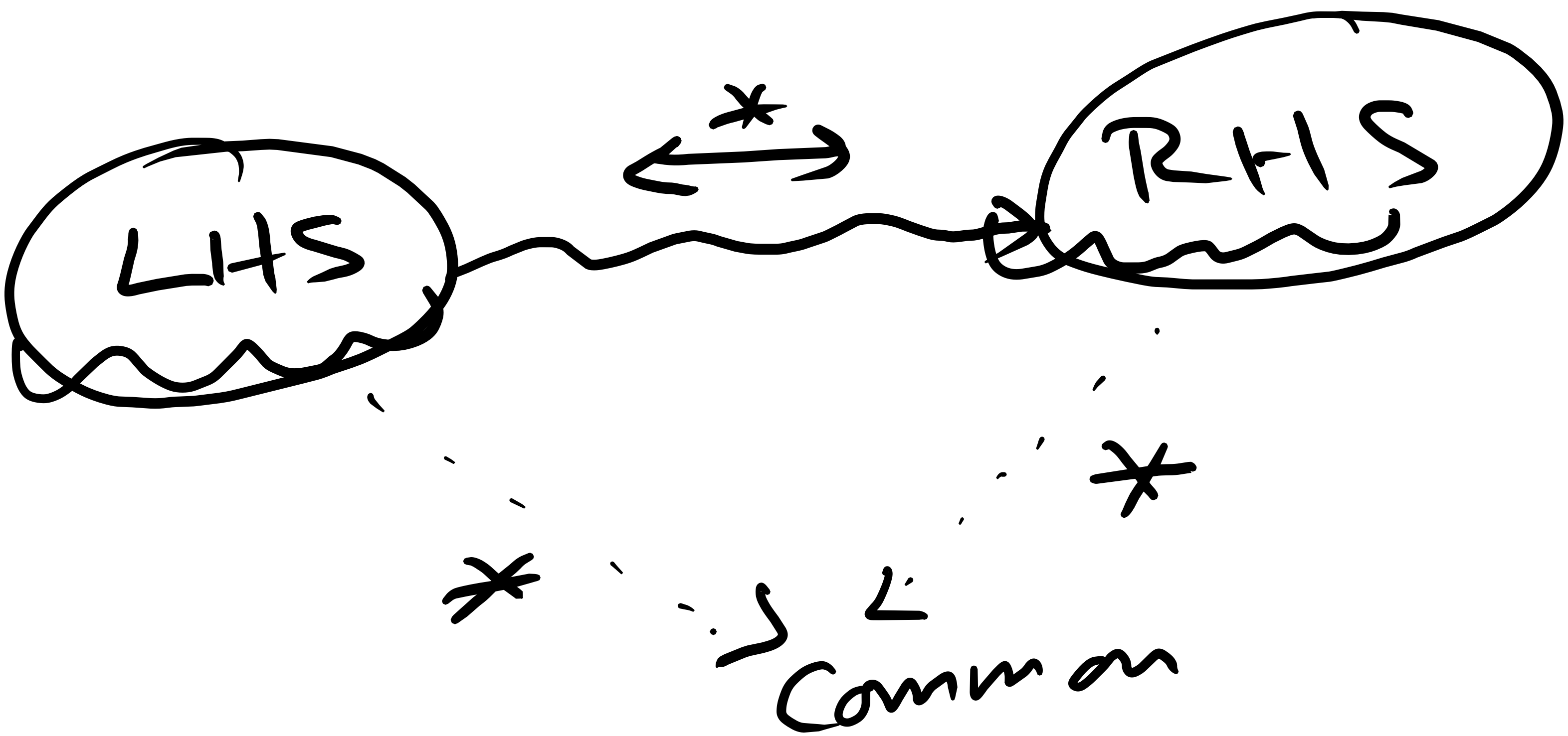
$$\begin{tikzcd}[column sep=small] & a \arrow[dl] \arrow[dr] & \\ b & & c \\ & ? & \end{tikzcd}$$

$b \overset{*}{\longleftrightarrow} c$ but

$b \not\downarrow c$

$$a * a^{-1} = 1$$

$$a * 1 = a$$



$A_2 : \longleftrightarrow$ $\qquad X_{A_2} = \mathbb{N} \setminus \{0, 1\}$

$x \xrightarrow{A_2} y :$ $\quad x = k \times y, \; k > 0$

$x \xleftrightarrow{A_2} y :$ either x divides y or y divides x.

A_2

$$
\begin{tikzcd}
 & & 24 \arrow[dl] \arrow[d] \arrow[dr] & \\
 & 6 \arrow[dl] \arrow[d] & 4 \arrow[dll, bend left] & 12 \arrow[dlll, bend left] \\
2 & 3 & &
\end{tikzcd}
$$

$x \xleftrightarrow[A_2]{*} y$

: interpretation

$x \xleftrightarrow[A_2]{*} y \overset{df}{=}$

$$\underline{X_{A_2} \times X_{A_2}}$$

$2 \xleftrightarrow[A_2]{*} 3$ but $2 \not\downarrow 3$