Lecture 2020-09-11-Fri

ADDITION Language:

<u>Syntax</u> & Interactive Semantics:

Syntax:

$$\text{Exp} \quad e ::= \bar{n} \mid e + e$$

$$+(e, e)$$

$$n \in \mathbb{N}$$

Inductive Terms over Σ

Σ: ?

$$\Sigma_{ADD} = \{+\} \cup \mathbb{N}$$

$$\alpha(n) = 0 \quad \alpha(+) = 2$$

Inductive Defn of ADDITION Expressions

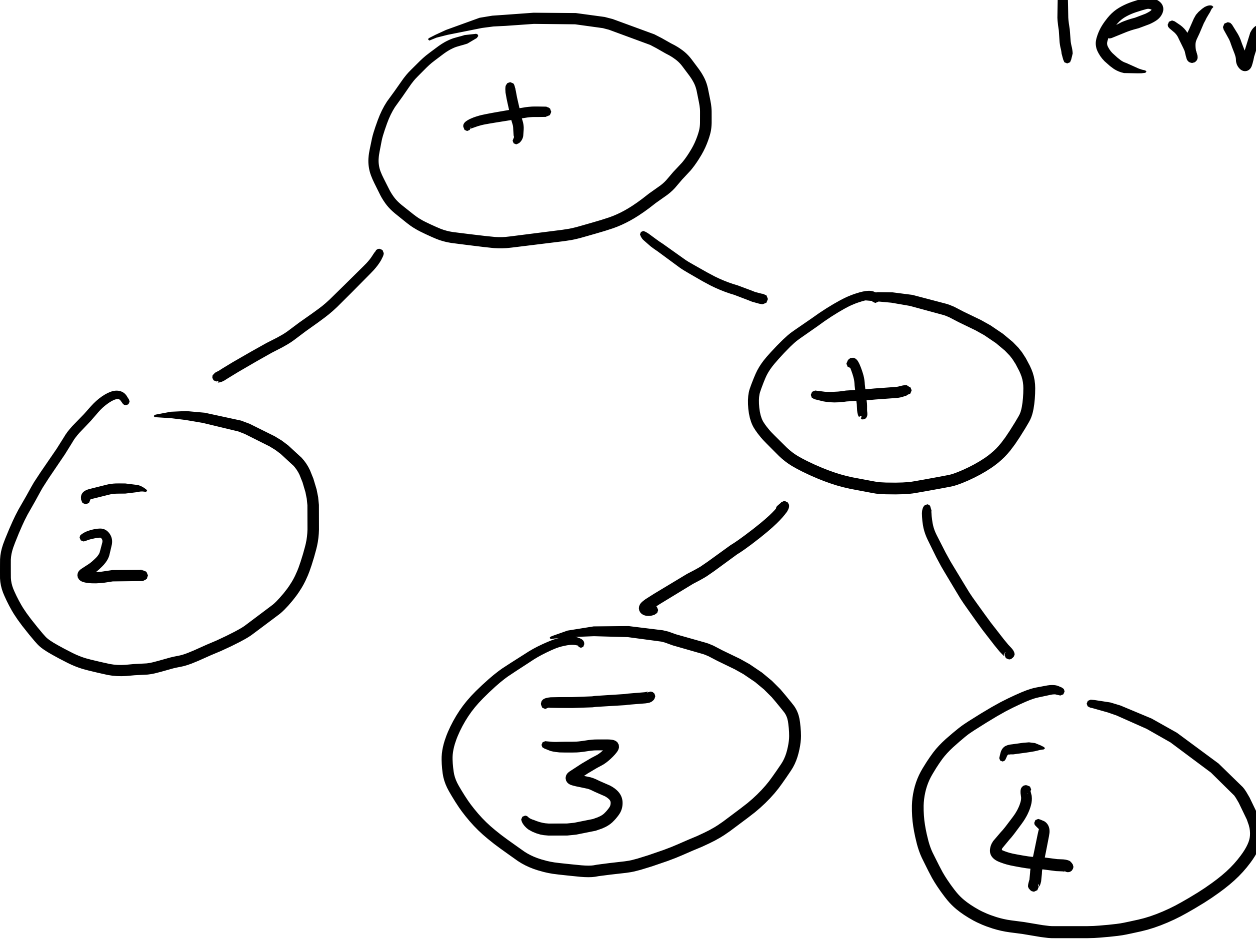
$$\dfrac{}{\bar{n}\ \ \mathrm{Exp}}\ n \in \mathbb{N} \quad \text{NUM}$$

$$\dfrac{e_1\ \mathrm{Exp} \quad e_2\ \mathrm{Exp}}{e_1 + e_2\ \mathrm{Exp}} \quad \text{PLUS}$$

Some obvious Facts:

1. $\bar{n}_1 = \bar{n}_2$ iff $n_1 = n_2$

2. $e_1 + e_2 = e_1' + e_2'$ iff $e_1 = e_1'$ and $e_2 = e_2'$



Term

From now, 'Term' will refer to inductive terms unless specified otherwise.

$$\left.\begin{array}{c} \bar{2} + (\bar{3} + \bar{4}) \\ \downarrow \\ \bar{2} + \bar{7} \longrightarrow \bar{9} \end{array}\right\} \text{Evaluation}$$

DYNAMICS & TRANSITION SYSTEMS

$$S = \langle X, X^0; U, \rightarrow, Y, h \rangle$$

X : state space

$X^0 \subseteq X$: set of initial states

U : action space

$\rightarrow \subseteq X \times U \times X$

$$x \xrightarrow[S]{u} x'$$

Y : Output space

$h : X \rightarrow Y$:

VIEW function

$x \xrightarrow[S]{} x'$ $=_{df}$:

$\exists u \in U$ s.t. $x \xrightarrow[S]{u} x'$

$x \xrightarrow[S]{u}$ $\quad$ $\exists x \in X$: s.t. $x \xrightarrow[S]{u} x'$

$x \xrightarrow[S]{}$ $\quad$ $\exists u, \exists x'$: $x \xrightarrow[S]{u} x'$

$x \not\xrightarrow[S]{}$ $\quad$ not $x \xrightarrow[S]{}$

x is <u>terminal</u> if $x \not\xrightarrow[S]{}$.

Run:

Finite Run:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{n-1}} x_n$$

Infinite Run

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \cdots$$

$$G_\Sigma := \langle V, hd : V \to \Sigma, D : V \times \mathbb{N} \to V \rangle$$

$X_a = V$

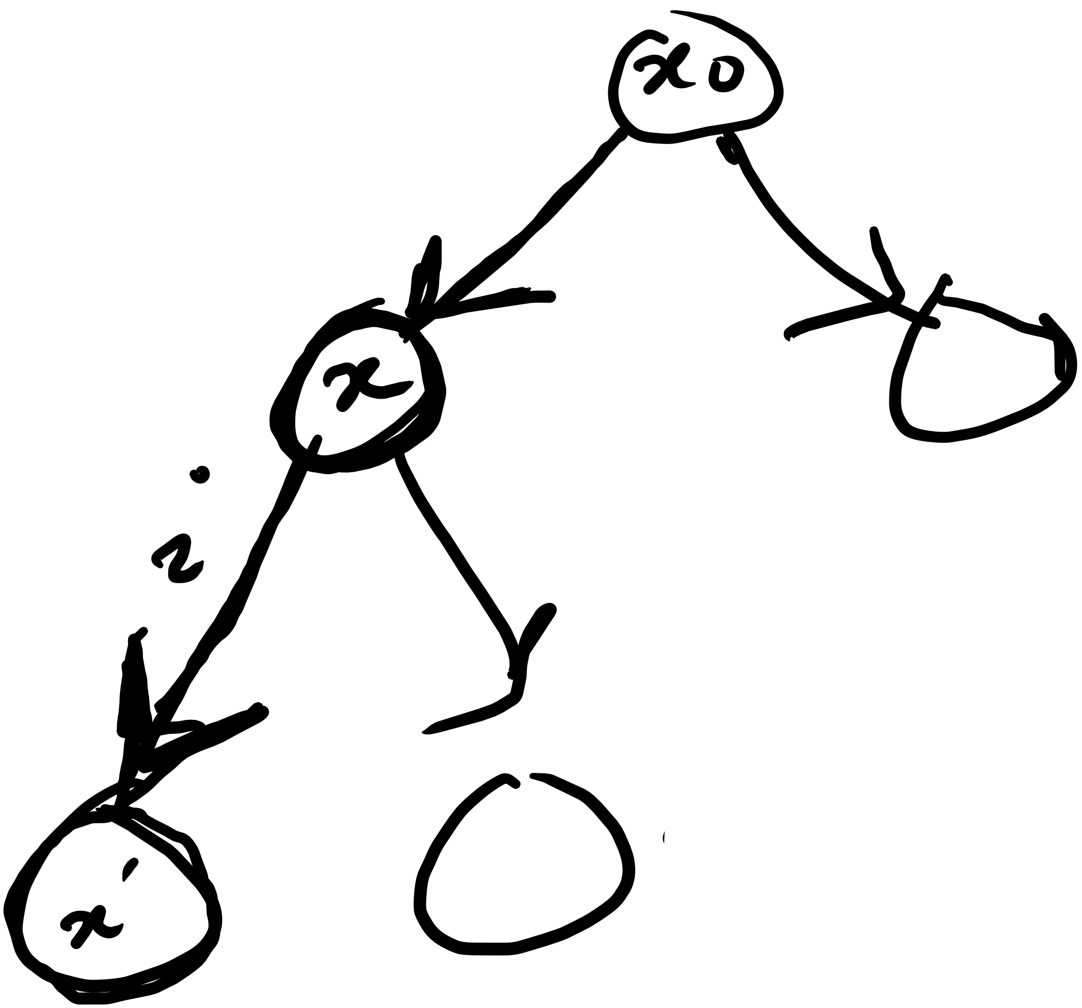
$X^0_a = X_a$

$U_a = \mathbb{N}$

$Y = \Sigma$

$h(x) = hd(x)$

$\longrightarrow = D$



x_0
x
z
x'

Terms, Positions & Subterms.

Let t be a term over Σ

let $p \in Pos(t)$

$t|_p =$ subterm of t at position p.

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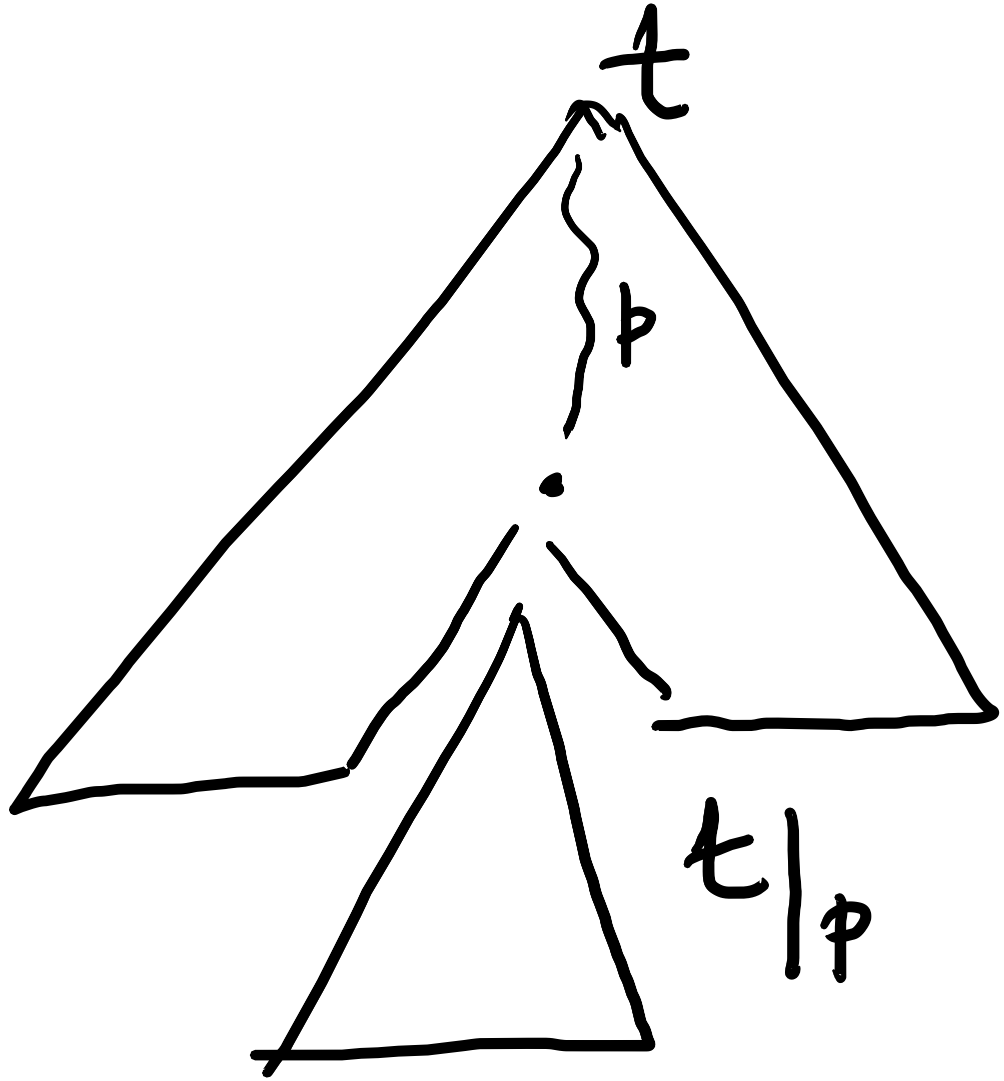
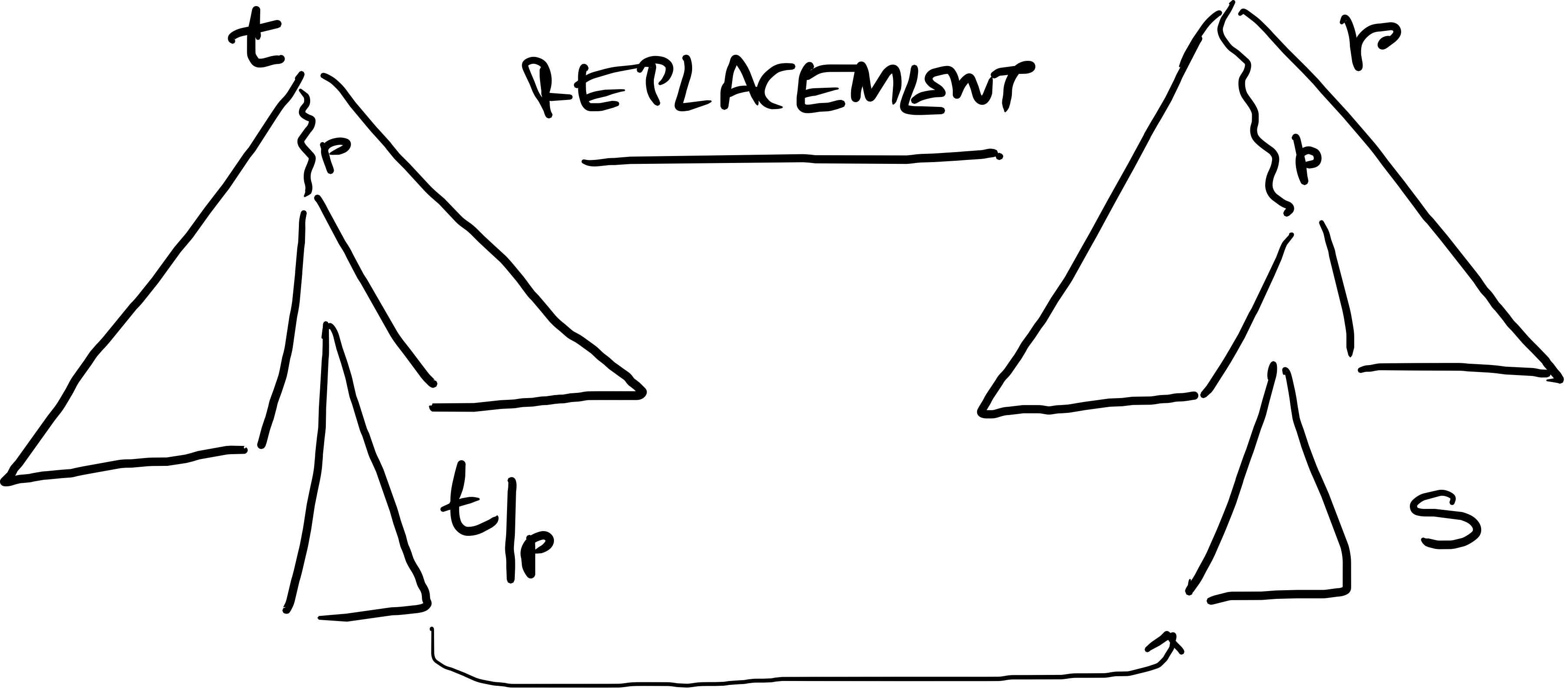


$$t = t_0 \xrightarrow{p_0} t_1 \rightarrow \cdots \quad t_n = t|_p$$

$p = p_0 p_1 \cdots p_{n-1}$

REPLACEMENT



$\mathrm{replace}(t, p, s)$

$\mathrm{replace} : [t : \mathrm{Term}, \mathrm{Pos}(t), \mathrm{Term}] \rightarrow \mathrm{Term}$

Rewrite Rule:

A ~~map~~ from Terms to Terms:

relation

$$e \hookrightarrow e'$$

For ADDITION

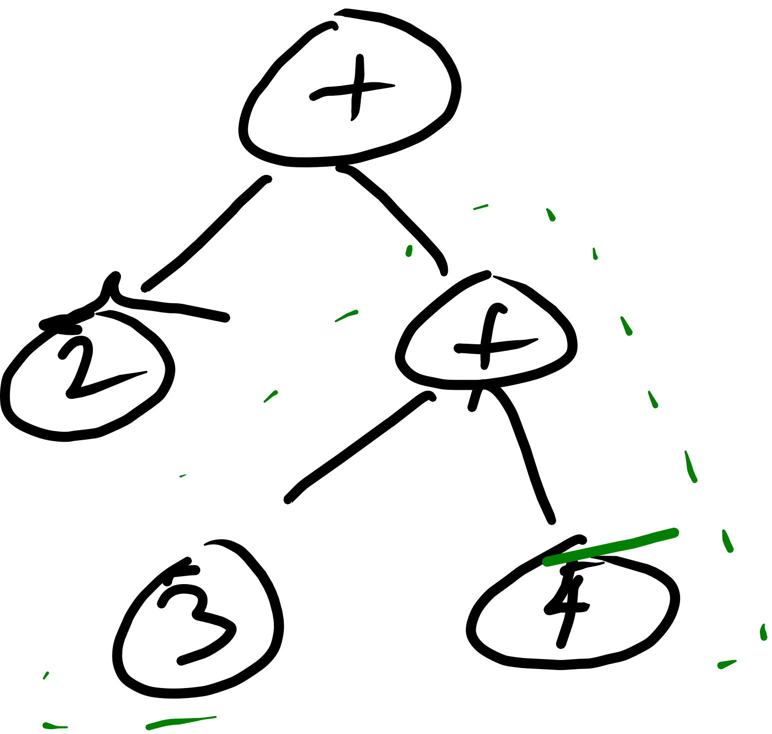
CONSTRUCTOR

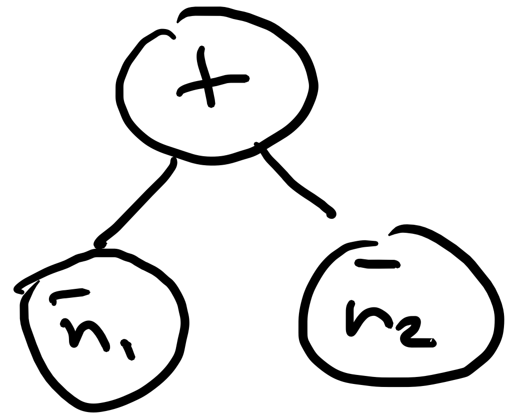
Rewrite :

$$\boxed{\overline{n_1} + \overline{n_2} \hookrightarrow \overline{n_1 + n_2}}$$

ADD RW

ADDITION operation





REDUCTION

Relation between Terms & Positions:

$$e \xrightarrow[ADD]{p} e'$$

$$S_{ADD}(\Sigma) = \left\langle X_{ADD}, \cancel{X^0_{ADD}}, U_{ADD}, \underset{ADD}{\longrightarrow}, \cancel{Y_{ADD}}, \cancel{h_{ADD}} \right\rangle$$

X_{ADD} = Expressions

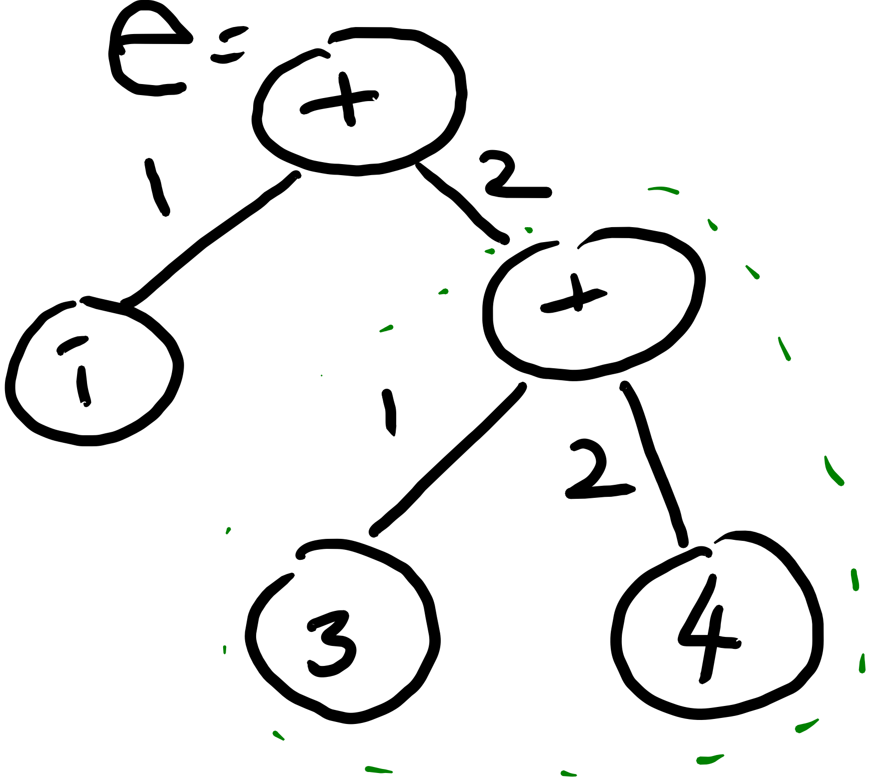
$$X^0_{ADD} = X_{ADD}$$

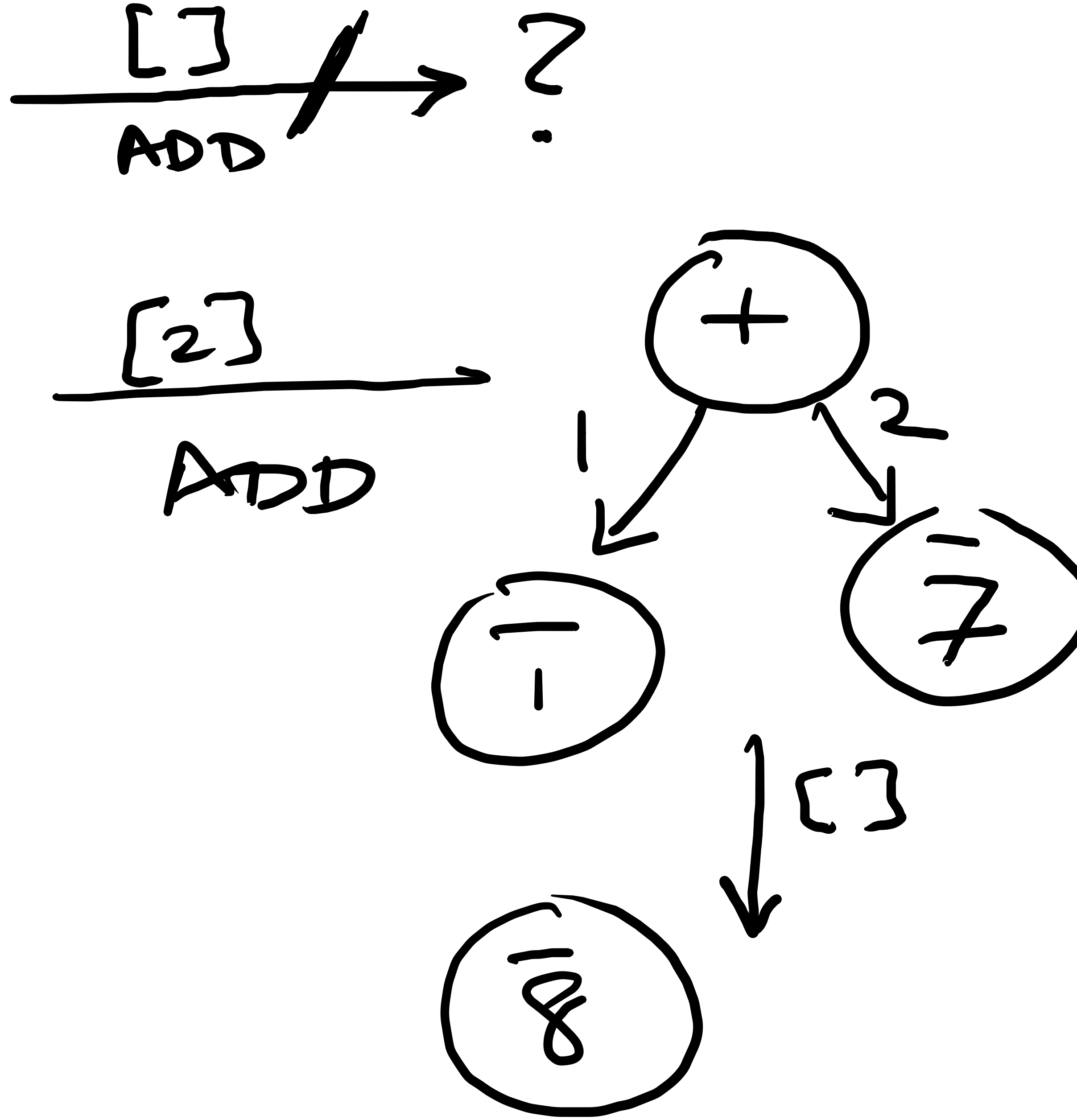
$U_{ADD} = \mathbb{N}^*$

$$e \xrightarrow[ADD]{p} e'$$

1. $e \in EXP$
2. $e' \in EXP$
3. $p \in Pos(e)$
4. $e|_p \hookrightarrow s \in ADD_{RW}$ for some s.
5. $e' = replace(e, p, s)$

e =





$e|_{[\,]} = e$

$e \xrightarrow[ADD]{} e'$ iff $\exists p \in \mathbb{N}^*$ s.t.

$$e \xrightarrow[ADD]{p} e'$$

$e \xrightarrow[ADD^*]{} e'$ if there is a finite run

$$e = e_0 \xrightarrow[ADD]{p_0} e_1 \xrightarrow[ADD]{p_1} \cdots \xrightarrow[ADD]{} e_n = e'$$

$$\bar{1} + (\bar{3} + \bar{4}) \xrightarrow{\text{ADD}} \bar{1} + \bar{7} \xrightarrow{\text{ADD}} \bar{8}$$

$$\bar{1} + (\bar{3} + \bar{4}) \xrightarrow[\text{ADD*}]{} 8$$

$$(\bar{2} + \bar{3}) + ((\bar{3} + \bar{6}) + \bar{7}) \xrightarrow[\text{ADD}]{} 5 + ((3+6)+7) \xrightarrow[\text{ADD}]{} 5 + (9+7) \xrightarrow[\text{ADD}]{} 21$$

$e_1 \hookrightarrow e_2$ Rewrite

$e \xrightarrow[\text{ADD}]{} e_2$ Reduce

$e \xrightarrow[\text{ADD}^*]{} e_2$ SIMPLIFY

$$e \overset{\quad}{\underset{ADD}{\not\longrightarrow}} \quad ? \text{ Possible?}$$

Proposition: $e \underset{ADD}{\not\longrightarrow}$ iff

$e \equiv \bar{n}$ for some $n \in \mathbb{N}$.

Proof: Exercise

Termination

There is no infinite run

$$e_0 \xrightarrow{p_0} e_1 \xrightarrow{p_2} \cdots\cdots e_n \xrightarrow{p_n} \cdots$$