ARITHMETIC

Syntax of expressions

$$\frac{n \in Num}{\bar{n} \in Exp}\ \text{NUM}$$

$$\frac{e_1 \in Exp \quad e_2 \in Exp}{e_1 + e_2 \in Exp}\ \text{PLUS}$$

$$Exp \ni e ::= \bar{n} \mid e_1 + e_2$$

$$n \in Num$$

Proof that $\bar{3} + \bar{5}$ is an expression

1. $3 \in Num$
2. $\bar{3} \in Exp$ 1, NUM
3. $5 \in Num$
4. $\bar{5} \in Exp$ 3, NUM
5. $\bar{3} + \bar{5} \in Exp$ 2, 4, PLUS

$$\text{PLUS}\;\dfrac{\text{Num}\;\dfrac{3 \in Num}{\bar{3} \in Exp} \qquad \dfrac{5 \in Num}{\bar{5} \in Exp}\;\text{NUM}}{\bar{3} + 5 \in Exp}$$

Size of an expression:

$|e| \in Nat$

$|\ | : Exp \longrightarrow Nat$

$$|\bar{n}| \stackrel{df}{=} 1$$

$$|e_1 + e_2| \stackrel{df}{=} 1 + |e_1| + |e_2|$$

Recap of system

$$S = \langle X, X^{\circ}, U, \longrightarrow, \cancel{Y}, \cancel{h} \rangle$$

where X : state space

$X^{\circ} \subseteq X$: set of initial states

U : Input space

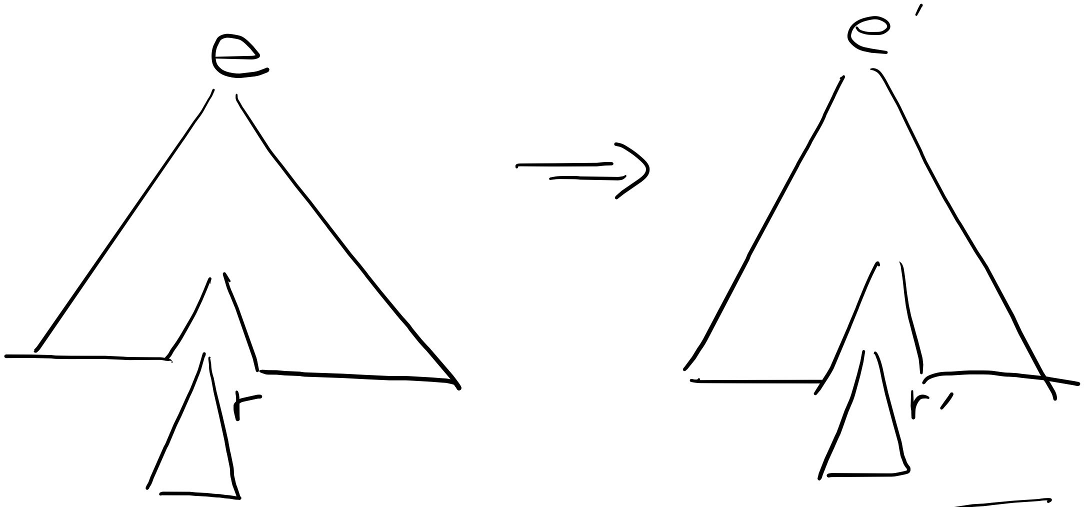
$\longrightarrow \subseteq X \times U \times X$ $\qquad x \xrightarrow{u} x'$

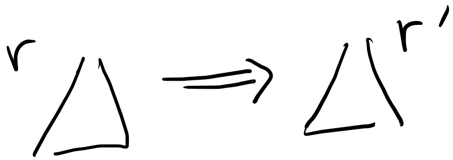
Y : output space

$h : X \longrightarrow Y$ view function

Assume $Y = X$ & $h = \mathrm{Id}_X$

Reduction (1-step):

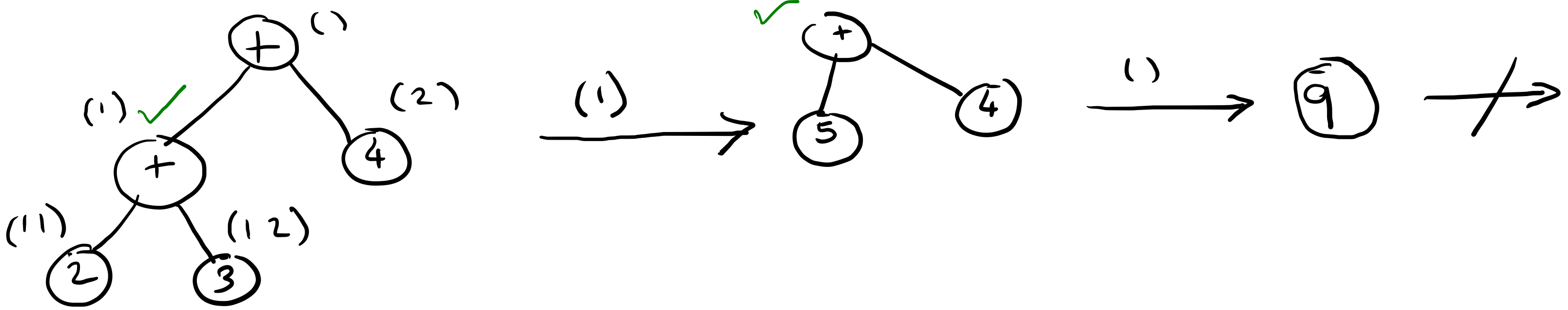




Rewrite Rule

$$\bar{n}_1 + \bar{n}_2 \Rightarrow \overline{n_1 + n_2}$$

A <u>redex</u> is a subexpression that matches the LHS of a rewrite rule.



$$X = Exp$$

$$X^{\circ} = X$$

$$U = (1+2)^{*} \quad (\text{regular expr})$$

$$e \xrightarrow{u} e' \quad =df$$

1) u is the position of a subexpression r of e.

2) r is a redex

3) $r \Longrightarrow r'$ using the rewrite rule

4) e' is the expression obtained by substituting r at posn u in e with r'

Semantics:

1. Termination:

$\forall e_0 \in Exp$: Every run starting from e_0 is of the form $e_0 \to e_1 \longrightarrow e_2 \cdots e_n \not\to$

$e_0 \to e_1 \longrightarrow e_2 \longrightarrow \cdots$ is a run if

$\exists u_0\ u_1 \cdots$

s.t. $e_0 \xrightarrow{u_0} e_1,\ e_1 \xrightarrow{u_1} e_2 \cdots$

Defn (Irreducible:)

e is <u>irreducible</u> if $\neg \exists e'$ st.

$$e \longrightarrow e'$$

Notation: $e \not\longrightarrow$

Defn $(e \longrightarrow e')$: $e \longrightarrow e' =df= \exists u : e \xrightarrow{u} e'$

Note that $e \not\longrightarrow$ implies $e \equiv \bar{n}$

Defn (NORMAL form): The expression $\bar{n}$ is called a <u>normal form</u>.

Termination:

For all $e_0 \in Exp$, either

$e_0 \equiv n$ in which case $e \not\rightarrow$, OR

$$e_0 \longrightarrow e_1 \longrightarrow \cdots\cdots \bar{n} \not\rightarrow$$

<u>Proof</u>: Based on the observation that

if $e \longrightarrow e'$, then $|e| > |e'|$

b) CONFLUENCE:

if $e_0 \longrightarrow e_1 \longrightarrow \cdots \bar{n}$

& $e_0 \longrightarrow e_1' \longrightarrow \cdots \bar{n}'$

then $n = n'$

SMALL-STEP operational semantics

We define a machine S where

$X = Exp = X^0$

$U = \{1\}$. Since U is a singleton, the transition relation $x \xrightarrow{1} x'$ will be written as $x \longrightarrow x'$

Now, let's define $\longrightarrow$

$$\frac{n_1 \in \mathrm{Num} \qquad n_2 \in \mathrm{Num}}{\bar{n}_1 + \bar{n}_2 \longrightarrow \overline{n_1 + n_2}}\ \text{NUM}$$

$$\frac{e_1 \longrightarrow e_1'}{e_1 + e_2 \longrightarrow e_1' + e_2}\ \text{LEFT}$$

$$\frac{e_2 \longrightarrow e_2'}{e_1 + e_2 \longrightarrow e_1 + e_2'}\ \text{RIGHT}$$

$$(\bar{2}+\bar{3})+\bar{4} \xrightarrow{P_1} \bar{5}+\bar{4} \xrightarrow{P_2} \bar{9}$$

$$P_1 \left\{ \dfrac{\dfrac{2 \in Num \qquad 3 \in Num}{\bar{2}+\bar{3} \longrightarrow \bar{5}}\ \text{NUM}}{(\bar{2}+\bar{3})+\bar{4} \longrightarrow \bar{5}+\bar{4}}\ \text{LEFT} \right.$$

$$P_2 \left\{ \dfrac{5 \in Num \qquad 4 \in Num}{\bar{5}+\bar{4} \longrightarrow \bar{9}} \right.$$

$$e_0 \longrightarrow e_1 \longrightarrow e_2 \longrightarrow e_3 \cdots$$

We now define another inductive system of transitions that captures runs.

Define $e \xrightarrow{+} e'$ inductively:

e reduces to e' in one or more steps of $\longrightarrow$.

$$\frac{e \longrightarrow e'}{e \xrightarrow{+} e'} \text{ 1-STEP}$$

$$\frac{e \longrightarrow e' \qquad e' \xrightarrow{+} e''}{e \xrightarrow{+} e''} \text{ TRANS}$$

$$e_0 \xrightarrow{+} e_n$$

$$e_0 \longrightarrow e_1 \longrightarrow \cdots\ e_n$$