Linear Algebra

Trivandrum School on Communication, Coding and Networking 2017

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Linear Algebra

- Vector Spaces
  - Definitions : Fields and Vector Space.
  - Linear Combinations.
  - Linear Independence and Dependence.
  - Subspaces
  - Basis and Dimension.
  - Vectors as tuples.
  - Basis change matrix.
- Linear Transformations.
  - Definition.
  - Linear Transformations as Matrices.
  - Similar matrices.
  - Range and Null Space of Linear Transformations.
  - Rank-Nullity Theorem.
  - Eigen values and vectors of a Linear Operator.

General ideas about Math-based Education and Research

- Math is not hard!
- There are only sets and maps (relations between sets).
- Start from basic axioms.
- Connect simple facts to create bigger facts (not always easy!).
- Imagination and Creativity.

Field - A rough definition

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- Space = set.
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Need for Linear Algebra in Communications and Coding

- For $\boldsymbol{x} = x(t)$, $\boldsymbol{y} = y(t)$ (complex-valued functions), define

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- If $x(t)$ is finite-energy, then so is $cx(t)$ for any $c \in \mathbb{C}$.
- To show : If $x(t), y(t)$ are finite-energy, then so is $x(t) + y(t)$.

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- To show : If $x(t), y(t)$ are finite-energy, then so is $x(t) + y(t)$.
- Given : $||\boldsymbol{x}|| < \infty$, $||\boldsymbol{y}|| < \infty$, show $||\boldsymbol{x} + \boldsymbol{y}|| < \infty$.

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$$\begin{aligned}||\boldsymbol{x}+\boldsymbol{y}||^2 &= ||\boldsymbol{x}||^2+||\boldsymbol{y}||^2+<\boldsymbol{x},\boldsymbol{y}>+<\boldsymbol{y},\boldsymbol{x}> \\ &\leq ||\boldsymbol{x}||^2+||\boldsymbol{y}||^2+2|<\boldsymbol{x},\boldsymbol{y}>|. \\ &\leq ||\boldsymbol{x}||^2+||\boldsymbol{y}||^2+2||\boldsymbol{x}||.||\boldsymbol{y}|| \quad (\text{if } \ |<\boldsymbol{x},\boldsymbol{y}>| \leq ||\boldsymbol{x}||.||\boldsymbol{y}||) \\ &< \infty \qquad (\text{as each of the above terms are finite})\end{aligned}$$

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Cauchy-Schwarz inequality

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|<\boldsymbol{x},\boldsymbol{y}>| \leq ||\boldsymbol{x}||.||\boldsymbol{y}||
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Proof: Fact: $||\boldsymbol{x}-\lambda\boldsymbol{y}||^2 \geq 0$, for any $\lambda \in \mathbb{C}$. Expand this and substitute $\lambda = \frac{<\boldsymbol{x},\boldsymbol{y}>}{||\boldsymbol{y}||^2}$.

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(Turns out that $<x,y>$ is also an example of a linear algebraic object called inner product)

Need for Linear Algebra in Communications and Coding

1. Finite-energy signals form a vector space over $\mathbb{C}$.

Field : Formal Definition

Definition: Fields

A *field* $\mathbb{F}$ is a set S with two operations (addition $(+)$ and multiplication(.)) such that

- For any $a, b \in S$, $a + b \in S$ (*closure under addition*)

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- For all $a, b \in S$, $a + b = b + a$ (*Addition is Commutative*)

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Definition: Fields (continued)

..such that..

- S is closed under multiplication.
- Multiplication is associative.
- Multiplicative identity exists (denoted by 1).
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Definition: Fields (continued)

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...such that..

- For all $a, b, c \in S$, $a.(b + c) = a.b + a.c$ (Distributivity of multiplication).

It is really over! (I think)

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Fields: Informally

Fields

A set where we can add, multiply, subtract (add with additive inverses), and divide (multiply with multiplicative inverses) and things work out *nicely*.

- Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$.
- Non-examples: $\mathbb{R}^{m \times k}$ matrices ($m = k \neq 1$).

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- Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$.
- Non-examples: $\mathbb{R}^{m \times k}$ matrices ($m = k \neq 1$).
- Think: What kind of structure exist if $k = m = 1$?, $k = m$?, $k \neq m$?.

Vector Spaces : Formal Definition

A set V is a vector space over $\mathbb{F}$ *(field of scalars)* if the following properties are satisfied :

- V is closed under vector addition, which is commutative and associative. $\forall \boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v} \in V$.

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  $\forall \boldsymbol{x} \in V, \forall \alpha \in \mathbb{F}, \alpha \boldsymbol{x} \in V$.

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- $\forall \boldsymbol{x}, \boldsymbol{y} \in V$ and $\alpha, \beta \in \mathbb{F}$
  1. $1\boldsymbol{x} = \boldsymbol{x}$
  2. $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$
  3. $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x})$
  4. $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$

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Vector Space: Informal Definition

Vector space V over $\mathbb{F}$

A set closed under addition, scalar multiplication (multiplication by scalars from $\mathbb{F}$).

Notation:

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Notation:

- Normal font, (α, β) for scalars.
- Bold fonts $(\boldsymbol{v}, \boldsymbol{w})$ for vectors.
- Caps for Vector spaces (V, W).
- $\mathbb{F}$ for field.

Subspaces

- $W \subseteq V$ is called a subspace if it is a vector space (over $\mathbb{F}$).
- A subset of W is a subspace if and only if :
  - For all $\boldsymbol{v}, \boldsymbol{w} \in W$, $\alpha \boldsymbol{v} + \boldsymbol{w} \in W, \forall \alpha \in \mathbb{F}$.

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Linear Combination of vectors

- A linear combination of a set of vectors $S = \{\boldsymbol{v_i} : i = 1, ..., r\} \subset V$ is

$$\sum_{i=1}^{r} \alpha_i \boldsymbol{v_i},$$

for some $\alpha_i \in \mathbb{F}$.

- Note that if $\alpha_i = 0, \forall i$, then the linear combination gives the $\mathbf{0} \in V$.
- Examples: $S = \{(1\ 0\ 0), (0\ 1\ 0)\}$. Then $(1\ 1\ 0)$ is a linear combination.

Linear Dependence

Linear Dependence of vectors

- Vectors $\{\boldsymbol{v_i} : i = 1, ..., r\}$ are called *linearly dependent*

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  for some α_is, at least one of which is non-zero.
- If $\alpha_j \neq 0$ for some $1 \leq j \leq r$ then

$$\boldsymbol{v_j} = \sum_{i=1, i \neq j}^{r} \beta_i \boldsymbol{v_i},$$

  where $\beta_i = \frac{-\alpha_i}{\alpha_j}$.

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Linear Independence

- If $\{\boldsymbol{v_i} : i = 1, ..., r\}$ is not linearly dependent, then they are *linearly independent*.
- Only zero-linear combination gives $\mathbf{0}$.

Examples

- Consider the vectors (from $\mathbb{R}^2$)

$$S = \left\{ \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \tag{1}$$

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- Consider $S \cup \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. This is linearly dependent.
- Consider $S \cup \{\mathbf{0}\}$. This is linearly dependent.

Span of a subset of vectors

Span

The span of a set of vectors $S = \{\boldsymbol{v_i} : i = 1, ..., r\}$ is the set of all linear combinations of the vectors in that set.

$$span(S) = \left\{ \sum_{i=1}^{r} \alpha_i \boldsymbol{v_i} : \alpha_i \in \mathbb{F} \right\}.$$

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- Any set of k-linearly independent vectors of $\mathbb{F}^k$.

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Theorem

Any two bases for a subspace contain the same number of vectors

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7. But that means C is dependent (contradiction).

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The following are equivalent (Prove it!):

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Dimension of a Subspace W

$$dim(W) = \text{No. of vectors in any basis of } W.$$

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Basis Extension

Theorem

Let V be a finite dimensional vector space and S be a linearly independent subset of vectors from V. Then S can be extended to a basis of V, i.e., there is a basis B for V such that $S \subseteq B$.

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- We will have a basis for V at the end.

Vectors from n-dimensional V.S as n-tuples

Unique representation of vectors using basis vectors

Let V be a n-dimensional vector space with basis $B = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$. Then any vector $\boldsymbol{v} \in V$ can be written as a unique linear combination of the basis vectors

$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{b_i}.$$

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- In terms of the basis B, we can represent $\boldsymbol{v}$ as the n-tuple,

$$[\boldsymbol{v}]_B = (\alpha_1, \alpha_2, ..., \alpha_n).$$

- This is only a representation, and may change with the basis chosen.

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Vectors as coordinates

- Let $V = \mathbb{R}^2$. Let $B = \{\boldsymbol{b_1} = (1,0), \boldsymbol{b_2} = (0,1)\}$.
- Consider a vector $\boldsymbol{v} = (5,6)$.
- $\boldsymbol{v} = 5\boldsymbol{b}_1 + 6\boldsymbol{b}_2$.
- In terms of B, we have

$$[\boldsymbol{v}]_B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Change of Basis

How do vector-representations change with change in the basis (from $B = \{\boldsymbol{b_i} : i = 1..n\}$ to $C = \{\boldsymbol{c_i} : i = 1..n\}$) chosen?

$$\text{Given } [\boldsymbol{v}]_B, \text{what is } [\boldsymbol{v}]_C?$$

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Given $[\boldsymbol{v}]_B$, what is $[\boldsymbol{v}]_C$?

- Given $B = \{\boldsymbol{b_i}\}$, we have

$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{b_i},$$

how to get β_is such that

$$\boldsymbol{v} = \sum_{i=1}^{n} \beta_i \boldsymbol{c_i},$$

i.e. what is $[\boldsymbol{v}]_C$?

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Change of Basis

Note that

$$
\begin{aligned}
[\boldsymbol{v}]_C &= \sum_{i=1}^{n} \alpha_i [\boldsymbol{b_i}]_C. \\
&= \left[\; [\boldsymbol{b_1}]_C \; [\boldsymbol{b_2}]_C \; .... \; [\boldsymbol{b_n}]_C \; \right] \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \\
&= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}
\end{aligned}
$$

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Change of Basis

Note that

$$[\boldsymbol{v}]_C = \sum_{i=1}^{n} \alpha_i [\boldsymbol{b_i}]_C.$$

$$= \begin{bmatrix} [\boldsymbol{b_1}]_C & [\boldsymbol{b_2}]_C & \dots & [\boldsymbol{b_n}]_C \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$= \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$\begin{bmatrix} [\boldsymbol{b_1}]_C & [\boldsymbol{b_2}]_C & \dots & [\boldsymbol{b_n}]_C \end{bmatrix}$ is known as the basis change matrix.

Basis change : Example

- Consider the basis $C = \{\boldsymbol{c_1} = (1, 0), \boldsymbol{c_2} = (1, 1)\}$ for $\mathbb{R}^2$.
- Let $\boldsymbol{v} = (5, 6)$. What is $[\boldsymbol{v}]_C$?

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- Let $\boldsymbol{v} = (5,6)$. What is $[\boldsymbol{v}]_C$?
- $$\begin{aligned} [\boldsymbol{v}]_C &= 5[\boldsymbol{b}_1]_C + 6[\boldsymbol{b}_2]_C \\ &= 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 6 \end{bmatrix}. \end{aligned}$$
- Check : $\boldsymbol{v} = -1\boldsymbol{c_1} + 6\boldsymbol{c_2}$.

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Need for Linear Algebra in Communications and Coding

$\mathcal{L}$=Finite energy signals which are also time-limited from $[0, T]$.

Theorem

A basis for $\mathcal{L}$ is

$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \;\; i = 0, \pm 1, \pm 2, ...$$

.

Proof:

- Fourier Series expansion.

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Need for Linear Algebra in Communications and Coding

1. Finite-energy time-bounded signals form a vector space.

Need for Linear Algebra in Communications and Coding

1. Finite-energy time-bounded signals form a vector space.
2. Span of time-limited sinusoids = Time-limited Finite-Energy signals
   - The sinusoidal basis helps to easily characterize output signal when the signal is passed through 'linear time-invariant' systems.
   - Can think of signals as vectors. Makes Digital Communication possible!

Linear Transformations

- Maps between Vector Spaces (defined over a common field $\mathbb{F}$).
- We like linearity.

Linear Transformation

Let V and W be vector spaces over the field F. A function $T : V \to W$ is a linear transformation if

$$T(c\mathbf{v_1} + \mathbf{v_2}) = cT(\mathbf{v_1}) + T(\mathbf{v_2}), \forall \mathbf{v_1}, \mathbf{v_2} \in V, \text{and}, \forall c \in \mathbb{F}.$$

If $V = W$, then T is called a *linear operator*.

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Linear Tranformation : Examples and Non-Examples

1. T: $R^{2\times 2} \to R$ where T is defined as
   $T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = x_1 + x_4.$

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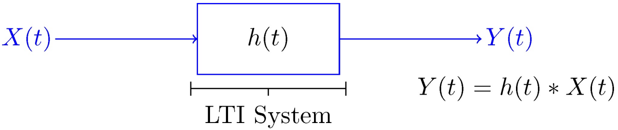
   (**No** if $a \neq 0$, **Yes** if $a = 0$)

Linear Transformation : Examples and Non-Examples



- $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$.
- Is this is a linear transformation? (What are its domain and codomain?)

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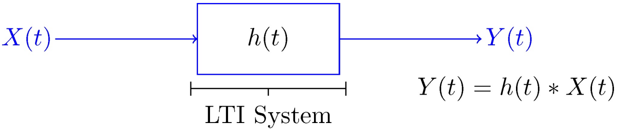

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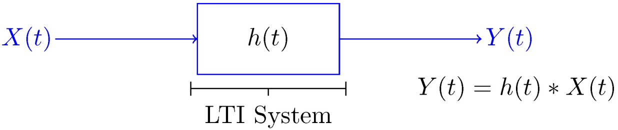

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- Linear Transformation.
- Domain=Codomain=Vector Space of Finite energy signals.

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Need for Linear Algebra in Communications and Coding

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2. Span of time-limited sinusoids = Time-limited Finite-Energy signals.
3. LTI systems are Linear Operators on the Space of Finite Energy Signals.

Sum and Composition of Linear Transformations

- T_1 and T_2 are linear transformations from $V \to W$. Then so is their 'sum' T defined as
$$T(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}).$$
- So is T' ('composition') defined as
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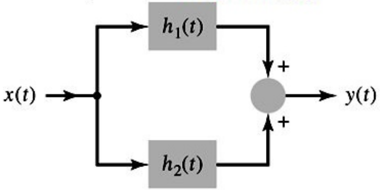
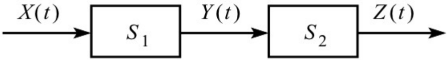
- Series and Parallel LTI systems.





Range and Null Space of a Linear Transformation

Range (Image) and Null-Space (Kernel) of T

- Range (Image):

$$R(T) = \{\boldsymbol{w} \in W : T(\boldsymbol{v}) = \boldsymbol{w}, \text{for some } \boldsymbol{v} \in V\}.$$

- Nullspace (kernel):

$$N(T) = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) = \mathbf{0} \in W\}.$$

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- $R(T)$ is a subspace of W.
- $N(T)$ is a subspace of V.

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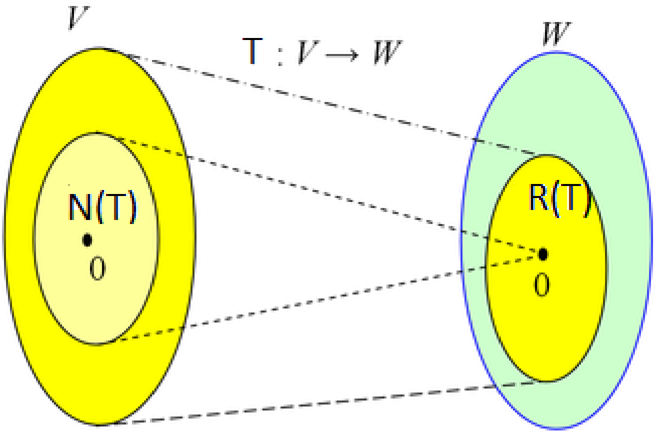
Range and Null Space



Rank Nullity Theorem

Rank and Nullity

- $Rank(T) = dim(R(T))$.
- $Nullity(T) = dim(N(T))$.

Rank Nullity Theorem

Let V be a finite dimensional vector space and $T : V \rightarrow W$ be a L.T. Then

$$dim(V) = Rank(T) + Nullity(T).$$

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Proof of Rank Nullity Theorem

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- It suffices to show that $\{T(\boldsymbol{v_{k+1}}), \dots, T(\boldsymbol{v_n})\}$ is a basis for $R(T)$.

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$$\begin{aligned}\mathbf{0} &= \sum_{i=k+1}^{n} \alpha_i T(\mathbf{v_{k+i}}) \\ &= T(\sum_{i=k+1}^{n} \alpha_i \mathbf{v_i}).\end{aligned}$$

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- This means $\sum_{i=1}^{n-k} \alpha_i \boldsymbol{v_{k+i}} \in N(T)$. Thus,

$$\sum_{i=k+1}^{n} \alpha_i \boldsymbol{v_i} = \sum_{i=1}^{k} \beta_i \boldsymbol{v_i}.$$

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Proof of Rank Nullity Theorem

- Rearranging,

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- This is a contradiction as $\{\boldsymbol{v_i} : i = 1, ..., n\}$ is a basis.
- Thus $\{T(\boldsymbol{v_{k+1}}), \ldots, T(\boldsymbol{v_n})\}$ is linearly independent.

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- Apply T on both sides to get the result.

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Example

- Let
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$
- Consider the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{x} \to A\boldsymbol{x}$.
- What is the $N(T)$? What is $R(T)$?
- Check if R-N theorem is satisfied.

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Matrix of a Linear Transformation

Characterising linear transformations

Theorem

Let $T : V \to W$ be a L.T. Let $B = \{\mathbf{v_i} : i = 1.., n\}$. Then the action of T on any arbitrary $\mathbf{v} \in V$ is completely specified by its action on the basis vectors $\{\mathbf{v}_i : i = 1, .., n\}$.

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- Choosing a basis B_W for W enables us to write $\boldsymbol{w}$ as a m-tuple $[\boldsymbol{w}]_{B_W}$.
- Fixing B_V and B_W, we have a matrix representation $[T]$ for T.

$$[T][\boldsymbol{v}]_{B_V} = [\boldsymbol{w}]_{B_W}$$

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Matrix of a Linear Transformation

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- $$i^{th} \text{ column of } [T] = [T(\boldsymbol{v_i})]_{B_W}.$$

Example

- Consider the Lin. Operator on the space of real polynomials of degree upto 2, defined as follows.

  $$T(a_0 + a_1 t + a_2 t^2) = (a_0 + a_2) + (a_1 + a_2)t + (a_0 + 2a_1 + 3a_2)t^2.$$

- Find its representation under (a) Basis $B = \{1, t, t^2\}$ (b) Basis $C = (1 + t, 1 + t^2, 1 + t + t^2)$.

Need for Linear Algebra in Communications and Coding

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    - Embed a low-D subspace in a High-D vector space to a Low-D vector space. (Compression or Source Coding)
    - Embed a low-D vector space as a Low-D subspace of a High-D vector space (Channel Coding).

Eigen values and vectors of a linear operator

Let $T : V \to V$ be a Linear Operator.

Eigen values and vectors

A non-zero $\boldsymbol{v} \in V$ and a constant $\lambda \in \mathbb{F}$ are called the eigen vector and its eigen value of T if

$$T(\boldsymbol{v}) = \lambda \boldsymbol{v}.$$

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- For certain types of Lin. Operators, there exists a basis $B = \{\boldsymbol{v}_i\}$ for V consisting of eigen vectors (with eigen values λ_is).

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- Understanding the I/O relationships of such Lin Operators are easy with such a basis.
- $$\begin{aligned} T(\mathbf{v}) &= T\left(\sum \alpha_i \mathbf{v}_i\right) \\ &= \sum \alpha_i T(\mathbf{v}_i) \\ &= \sum \alpha_i \lambda_i \mathbf{v}_i \end{aligned}$$

Example for Eigen vectors and Values

- $\mathcal{L}$=Finite energy signals which are also time-limited from $[0, T]$.
- A basis for $\mathcal{L}$ is

$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \;\; i = 0, \pm 1, \pm 2, \ldots$$

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$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \ \ i = 0, \pm 1, \pm 2, ...$$

- The function $f_i(t)$ are the eigen vectors for any LTI system given by L, with eigen value being the fourier series coefficient of $h(t)$ at $2\pi i/T$.

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Need for Linear Algebra in Communications and Coding

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1. Finite-energy time-bounded signals form a vector space.
2. Span of time-limited sinusoids = Time-limited Finite-Energy signals.
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4. Linear Transformations are heavily used in Coding Theory and Cryptography.
5. Fourier basis are also eigen vectors of LTI systems. So understanding I/O relationships of LTI systems is easy.

Thank You