Index Coding

Trivandrum School on Communication, Coding and Networking 2017

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International Institute of Information Technology, Hyderabad

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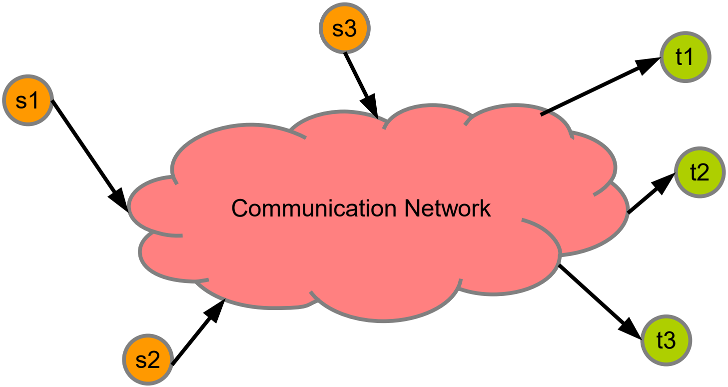
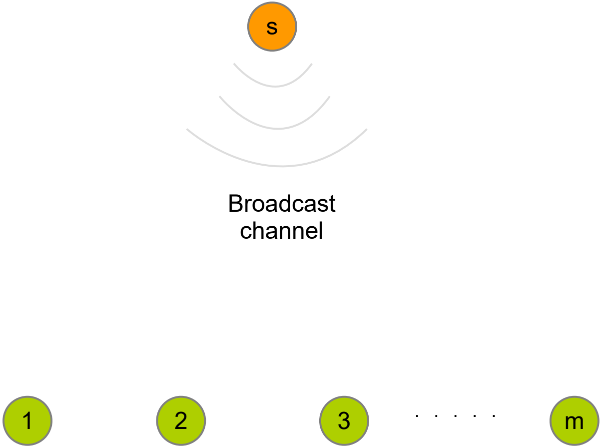
Multiuser Communication



Broadcast channel



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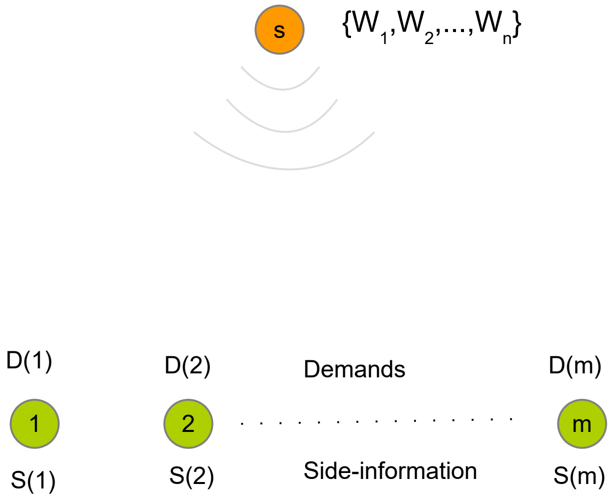
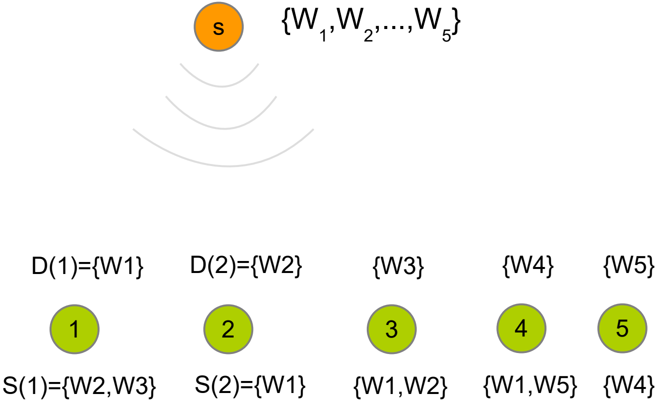
Index Coding Problem Setup

- A noiseless broadcast channel between the source and T receivers
- Messages $\mathcal{W} = \{W_i \in \mathbb{F}^t, i \in [1:n]\}$.
- Demand set at receiver j : $D(j) \subseteq \mathcal{W}$
- Side information at receiver j: $S(j) \subseteq \mathcal{W} \backslash D(j)$

Index Coding Problem Setup



Example problem



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Index Code: Definition

- Index code:
  Map from $\{t$-length message vectors$\} \rightarrow \{$l-length codewords$\}$
- $\mathbb{E} : \mathbb{F}^{nt} \rightarrow \mathbb{F}^{l}$

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Length and Transmission Rate

- Length of the Code $= l$.
- Transmission rate $= \frac{l}{n}$.

Want to find index codes of low transmission rate.

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Index Coding

- Multiuser Communication and Broadcast channels
- Index Coding Problem Setup
  - Definition of an Index Code (Encoding function).
  - Measures of interest
  - Scalar and Vector Linear Index Codes
- Two ways to think about Scalar Linear Index Coding
  - Fitting Matrix
  - Alignment approach
- An upper and lower bound on optimal length from graph theory.
- Some New and Open problems.

Measures of interest for an Index Code

- For an IC problem $\mathcal{I}$, define

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Goal of index coding (general)

Design codes that have transmission rate close to β.

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  - **Linear:** if the encoding function is linear
  - For a linear code, the encoding function is simply multiplication by a matrix G.

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  - **Vector:** if the messages are vectors $(t > 1)$.
  - **Scalar:** if the messages are scalars $(t = 1)$.

Length of a (scalar linear) index code

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- Optimal code $\Rightarrow$ **Minimum** number of linear combinations of info symbols such that every demand can be met.
  - *Focus for rest of the talk*: Scalar linear index codes.

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Two ways to think about Scalar Linear Index Coding

- Fitting matrix approach
- Interference alignment approach

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Two ways to think about Scalar Linear Index Coding

- Fitting matrix approach
- Interference alignment approach
- Variation is only in the method of code construction.
- After obtaining a code, we can 'look' at the code using both approaches.

Fitting matrix approach

- In general, for sink i can decoding for symbol W_j, it needs a linear combination of the form

$$W_j + \sum_{k \in S(i)} \alpha_k W_k,$$

(demanded symbol + some linear combination of side-info symbols).

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- In successful decoding, this linear combination is be obtained as a linear combination of the transmissions.
- For example (sink 1 to decode W_1) needs a linear combination :

$$W_1 + \alpha_2 W_2 + \alpha_3 W_3.$$

Fitting Matrix formulation

Define a matrix A with

- Number of rows $=$ Total number of demands at all sinks
- Number of messages $=n$.
- Every row corresponds to some demand (say symbol W_j) at some sink i.

Fitting Matrix formulation

The row corresponding to demand W_j at receiver i is filled with follows.

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- 'X's in the positions corresponding to $S(i)$.
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Fitting Matrix formulation

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- '1' in the j^{th} position (corresponding to W_j)
- 'X's in the positions corresponding to $S(i)$.
- 0s in other positions.
- For the example given before,

$$A = \begin{pmatrix} 1 & X & X & 0 & 0 \\ X & 1 & 0 & 0 & 0 \\ X & X & 1 & 0 & 0 \\ X & 0 & 0 & 1 & X \\ 0 & 0 & 0 & X & 1 \end{pmatrix}.$$

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Fitting Matrix formulation

- Note that we can fill the matrix A with any values for the Xs, and we get a solution (encoding function $\mathbb{F}^n \to \mathbb{F}^l$)

Fitting Matrix formulation

- Note that we can fill the matrix A with any values for the Xs, and we get a solution (encoding function $\mathbb{F}^n \rightarrow \mathbb{F}^l$)
- How to find the optimal code?
  - Find an assignment for the Xs which minimises the rank of A ($minrank(A) = l^*$)
  - Keep only l^* L.I rows, and throw away the rest.
  - The $l^* \times n$ matrix give an encoding matrix of an optimal code.

'Index Coding by Matrix completion'

Fitting Matrix formulation

- For the example:

$$A_{assigned} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Throw away row 3 and row 5.

Fitting Matrix formulation

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  Throw away row 3 and row 5.

- $$G_{opt} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

- This is infact an optimal code (we shall see why later).

Interference Alignment (IA) Formulation

- Linear index code:

$$G\boldsymbol{W} = \sum_{i=1}^{n} G_i W_i,$$

  where G_i is the i^{th} column of G.
- $G_i :=$ precoding vector for message W_i.

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where G_i is the i^{th} column of G.

- $G_i :=$ precoding vector for message W_i.
- Idea of interference alignment: To decode any demand, choose the interference (non-sideinformation) precoding vectors independent of the precoding vectors of the demanded symbol.

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IA approach - Another way to look at linear equation solving

- Let $\boldsymbol{W} = (W_1 \ W_2 \ W_3 \ W_4 \ W_5)^T, \boldsymbol{a} = (a_1 \ a_2 \ a_3 \ a_4)^T$.
- Let $G\boldsymbol{W} = \boldsymbol{a}$. We want $\boldsymbol{W}$.

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- Let $G\boldsymbol{W} = \boldsymbol{a}$. We want $\boldsymbol{W}$.
- Consider the columns of matrix G as $G_i, i = 1, 2, 3, 4, 5$.
- The given equations are in the form:

$$\sum_{i=1}^{5} W_i G_i = \boldsymbol{a}$$

.

- We can solve for $\boldsymbol{W}$, if G_is are independent.

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- We can thus obtain

$$\sum_{i=1}^{5} W_i G_i - \sum_{i \in S(j)} W_i G_i$$

$$= \sum_{i \notin S(j)} W_i G_i.$$

$$= \sum_{i \notin S(j) \cup k} W_i G_i + W_k G_k$$

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- W_k is decodable if G_k is independent of $\{G_i : i \notin S(j) \cup k\}$.

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- For example, consider decoding W_1 at receiver 1. For some index code matrix G, we can get

$$G_1 W_1 + (G_4 W_4 + G_5 W_5)$$

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IA Index Coding formulation

Design encoding matrix G such that

- G_k is independent of $G_i : i \notin S(j) \cup \{k\}$ for any demand k at any receiver j.
- G has the least number of rows.

Single-Unicast

- General Fitting Matrix/IA problem solving is hard.
- Bounds on Minimal Index Coding length exist for special classes.

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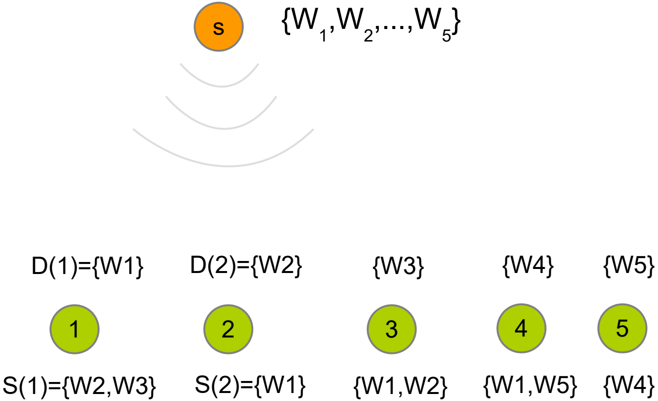
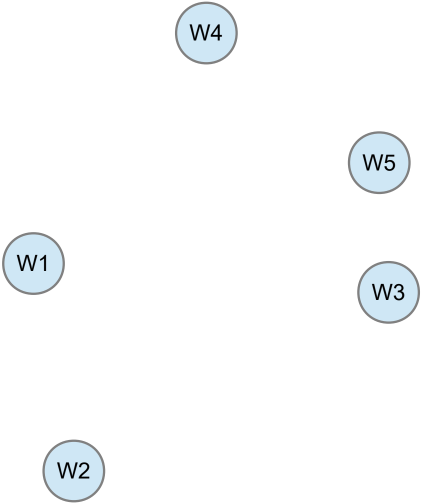
Single-Unicast

- Each receiver demands an unique message symbol.
- Can be represented by a ‘Side-information’ graph
  - SI graph has the vertices as the symbols (or receivers).
  - Edges indicate side-information available at the receivers.

Unicast example with SI graph



How to get the SI graph?- Vertices are messages



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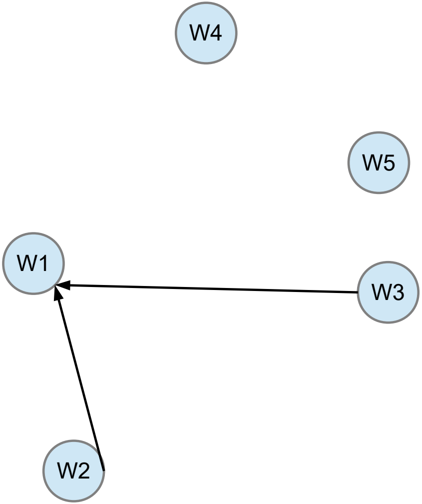
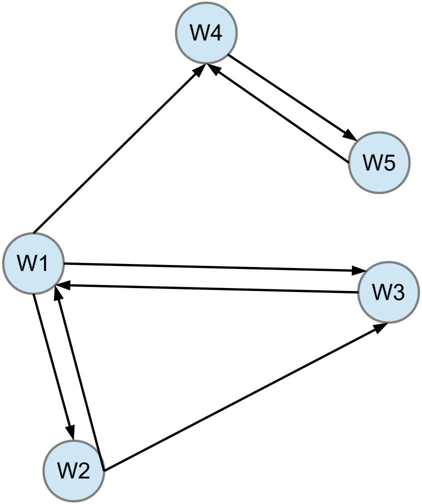
How to get the SI graph? - Edges indicate SI



Full SI graph of the example



Using the SI graph (and its complement) to obtain bounds

- Bounds from graph theoretic parameters.
- Upper bounds : Clique Cover, Chromatic Number, Local Chromatic Number, Cycle packing (Maximum number of edge-disjoint cycles).
- Lower bounds: Size of Maximum Acyclic Induced Subgraph (MAIS), Independence Number.

Upper bound from Cycle Packing

- Naive transmission is to send all n symbols uncoded.

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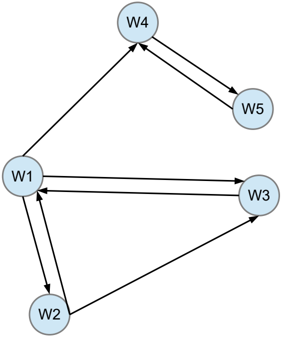
- Suppose there is a cycle in the SI graph consisting of r messages.
- Then we can satisfy these r-demands by $r - 1$ transmissions.



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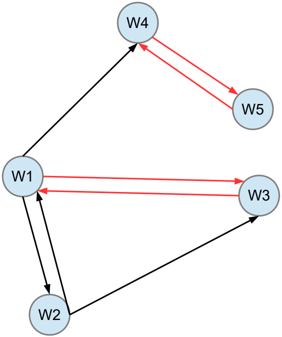
Upper bound from Cycle Packing

- Find the maximum number of vertex-disjoint cycles in the SI graph.
- For each cycle, we can 'save' one transmission.
- For the symbols not on any of the picked cycles, send uncoded symbols.



Lower bound : MAIS

Definition

Induced subgraph of SI graph $\mathcal{G}_{SI}$:
Pick a subset of vertices of $\mathcal{G}_{SI}$ and take **all** edges between them.

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Theorem

(MAIS lower bound:)
The length of the optimal code is at least as large as the size (number of vertices) in an maximum, acyclic, induced subgraph of $\mathcal{G}_{SI}$.

$$l^* \geq |MAIS|.$$

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- How to get MAIS? - Hard problem.
- We can get an AIS (not necessarily maximal) by deleting some vertices and seeing if the graph is acyclic ($\rightarrow$ this gives a lower bound)

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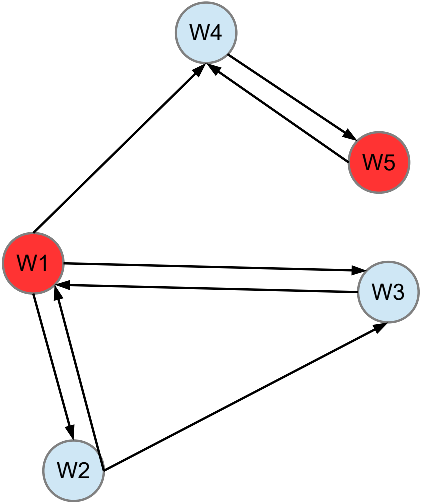
MAIS Lower Bound



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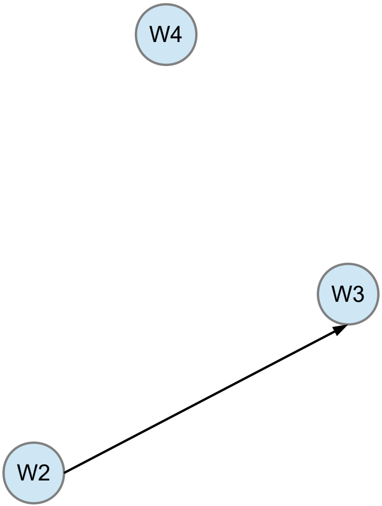

References

- Y. Birk and T. Kol, "Coding on Demand by an Informed Source (ISCOD) for Efficient Broadcast of Different Supplemental Data to Caching Clients", IEEE Transactions on Information Theory, Vol. 52, No. 6, June, 2006
- Z.Bar-Yossef, Y. Birk, T.S. Jayram, T. Kol, "Index Coding with Side Information", IEEE Transactions on Information Theory, Vol. 57, No. 3, March 2011.

New and Open Problems

Some New and Open problems.

- Pliable Index Coding
- Distributed Index Coding
- Connections to Distributed Storage.
- Field size for index coding.
- ⋮

Pliable Index Coding

- Receivers are happy to receive any message, rather than precisely asking for one.
- Probabilistic guarantees on length of Index Code based on some randomized algorithms.

"Pliable Index Coding", IEEE Trans. on. Info. Theory, 2015, S. Brahma and C. Fragouli (UCLA)

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Distributed Index Coding

- Multisource index coding.
- Information theoretic results on Capacity region.
- Graph theoretic upper bounds on special class of problems.

- "Distributed Index Coding", P. Sadeghi, F. Arbabjolfaei, Y.H. Kim, April 2016 (Australian National University and UCSD).
- "The Single-Uniprior Index-Coding Problem: The Single-Sender Case and the Multi-Sender Extension", L. Ong et al (University of NewCastle Australia, Institute of Infocomm research Singapore, IBM Research Singapore)

Connections to Distributed Storage

- Index Coding and Distributed Storage are dual-problems (in a sense).
- IC matrix G can be thought as the parity check matrix of a generator matrix of a Distributed Storage Code.

"On a Duality Between Recoverable Distributed Storage and Index Coding", Arya Mazumdar, March 2014 (University of Massachusetts at Amherst)

Field sizes for index coding

- Larger fields means more complexity.
- Want solutions over small fields

"Optimal solution for the index coding problem using network coding over GF(2)", J. Qureshi et al, Nanyang Tech Univ Singapore, June 2012

Thank You