# Rate $\frac{1}{3}$ Index Coding : Forbidden and Feasible Configurations

Prasad Krishnan

Joint work with Lalitha V
International Institute of Information Technology, Hyderabad

June 25-30, ISIT 2017

IIIT, HYDERABAD

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**WARNING: Talk contains graphic images.**

# Outline

1. Groupcast Index Coding from the Interference Alignment perspective
2. Known results for groupcast
3. Forbidden Configurations for Rate $\frac{1}{3}$ (Necessary Conditions)
4. Rate $\frac{1}{3}$ feasible configurations (Sufficient condition)

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# Groupcast index coding

- A broadcast channel between the source and $T$ receivers
- Messages $\mathcal{W} = \{W_i \in \mathbb{F}, i \in [1:n]\}$.
- Demand set at receiver $j$ : $D(j) \subseteq \mathcal{W}$
- Side information at receiver $j$: $S(j) \subseteq \mathcal{W} \backslash D(j)$

## Linear Index Code and its Rate

- Index code:
  Map from $\{$Messages$\} \rightarrow \{$l-length codewords$\}$
- Rate $R = \frac{1}{l}$.

# Interference alignment framework for index codes

- Index Coding map be $B_{l\times n}$; Transmitted vector is $B\boldsymbol{W}$.
- Consider a sink $j$ which demands message $W_k$.
- Sink $j$ can cancel the contributions from $S(j)$, obtaining

$$\sum_{i:i\notin S(j)} W_i \boldsymbol{b_i} = W_k \boldsymbol{b_k} + \sum_{i:i\notin S(j)\cup\{k\}} W_i \boldsymbol{b_i}$$

- Let $I(j,k) = \{W_i : i \notin S(j) \cup \{W_k\}\}$.
- Decoding is possible if $\boldsymbol{b_k}$ is independent of space spanned by vectors assigned to $I(j,k)$ (interference constraints).
- Choose matrix $B$ such that this is satisfied with least possible $l$ (alignment opportunities).

# Outline

# Known results for groupcast

- Tehrani et al, ISIT 2012, “Bipartite Index Coding”.
- Maleki et al, “Index Coding - An Interference Alignment Perspective”, ISIT 2012, TIT Sep. 2014.

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- via *alignment* and *conflict* graphs - polynomial complexity algorithm succeeds with high prob (large field).

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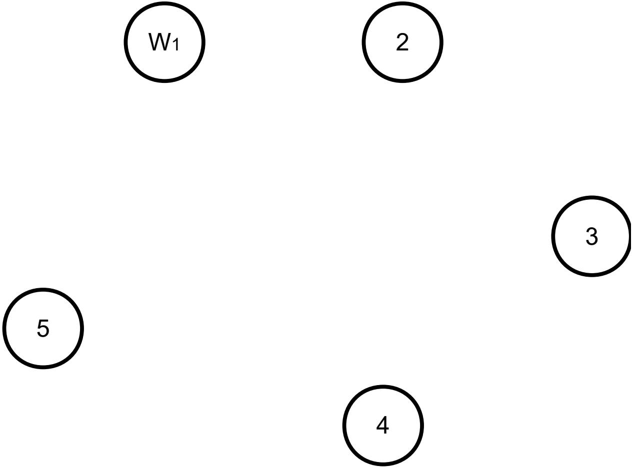
**Current results**

Subsume all known results on necesary and sufficient conditions for rate 1/3.

# Alignment graph and conflict hypergraphs (ISIT 2016)



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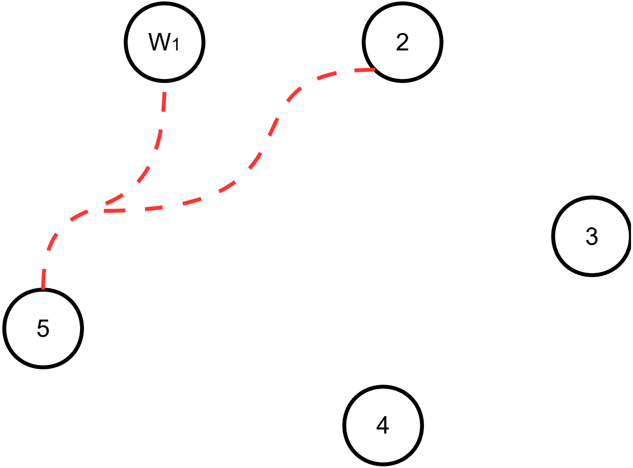

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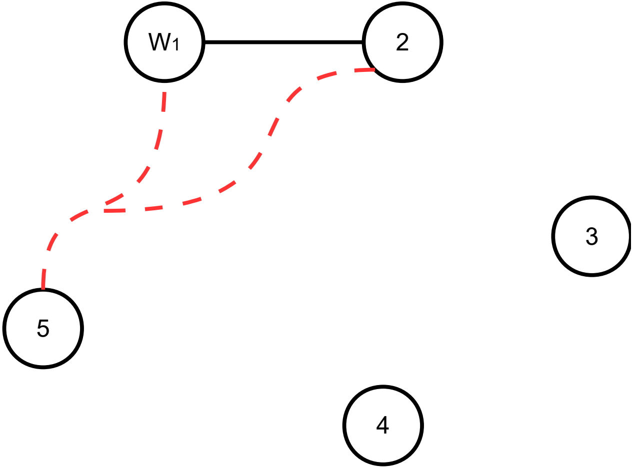

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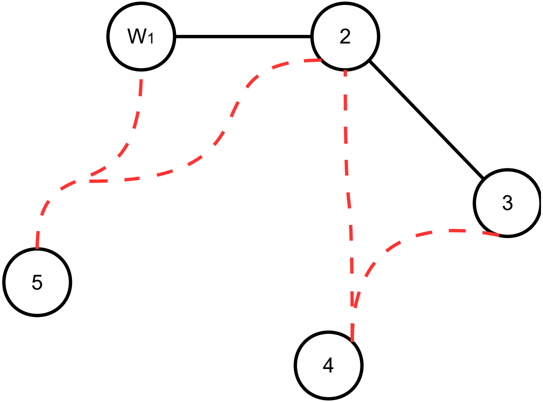
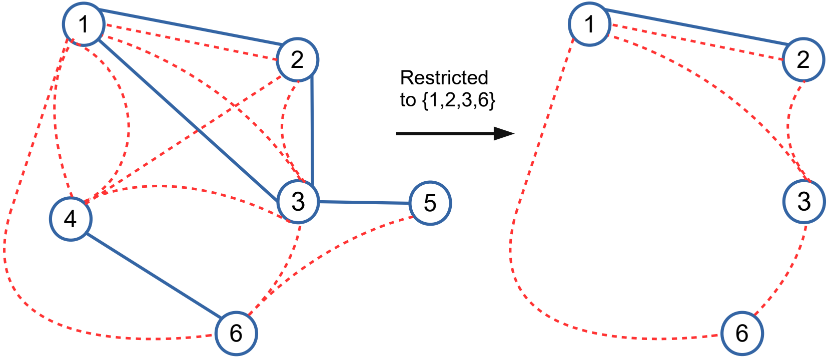



Hyperedge $\{W_5, \{W_1, W_2\}\}$ means

- $\{W_1, W_2\}$ *interfere* at (a receiver demanding) $W_5$.
- $\{W_1, W_2\}$ and $W_5$ are *in conflict*.

# Restricted IC problem and restricted conflicts (ISIT 2016)

- For $\mathcal{W}' \subset \mathcal{W}$, the **IC problem restricted to $\mathcal{W}'$** considers all demands and side-information only within $\mathcal{W}'$ at receivers.
- Restricted alignment graphs, Restricted conflict hypergraphs.



# Outline

# Strictly Rate $\frac{1}{L}$ subsets

Let $\mathbb{I}$ be an index coding problem with message set $\mathcal{W}$.

**Strictly Rate $\frac{1}{L}$ Subset**

A subset $\mathcal{W}' \subseteq \mathcal{W}$ is called Strictly Rate $\frac{1}{L}$ if

- in any rate $\frac{1}{3}$ code (if it exists),

  dim(space of vectors allocated to $\mathcal{W}'$)$= L$.

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**Rate 1/3 necessary condition**: The Restricted-IC problem to a strictly rate 1/2 set must have a rate 1/2 solution. (poly time to check by [Jaf]).

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## Strictly Rate $\frac{1}{L}$ Infeasible Subset

A subset $\mathcal{W}' \subseteq \mathcal{W}$ is called Strictly Rate $\frac{1}{L}$ **infeasible** if

- in any rate $\frac{1}{3}$ code, dim(space of vectors allocated to $\mathcal{W}'$)$\neq L$.

# Strictly rate $1/2$ subsets - A Key Idea

**Generating Large Strictly Rate $1/2$ sets from Smaller ones**

# Strictly rate 1/2 subsets - A Key Idea

### Generating Large Strictly Rate 1/2 sets from Smaller ones

Let $\mathcal{W}_i : i = 1, 2, ..., r$ $(r \geq 2)$ be strictly rate $\frac{1}{2}$ sets of an index coding problem $\mathbb{I}$ with message set $\mathcal{W}$, such that the sets

- $\mathcal{W}_1 \cap \mathcal{W}_2$
- $(\mathcal{W}_1 \cup \mathcal{W}_2) \cap \mathcal{W}_3$
- $\ldots$
- $\left(\cup_{i=1}^{r-1} \mathcal{W}_i\right) \bigcap \mathcal{W}_r$.

are strictly rate 1 infeasible. Then the set $\cup_{i=1}^{r} \mathcal{W}_i$ must be strictly rate $\frac{1}{2}$.

# Examples of Strictly Rate 1/2 Configurations

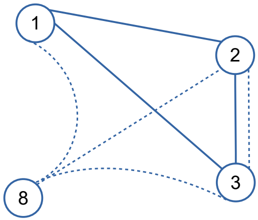


Figure: Triangular Interfering Set

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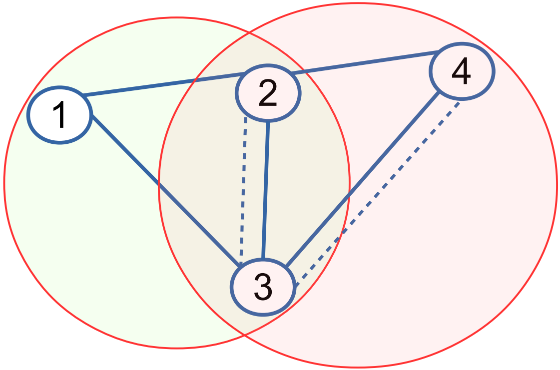

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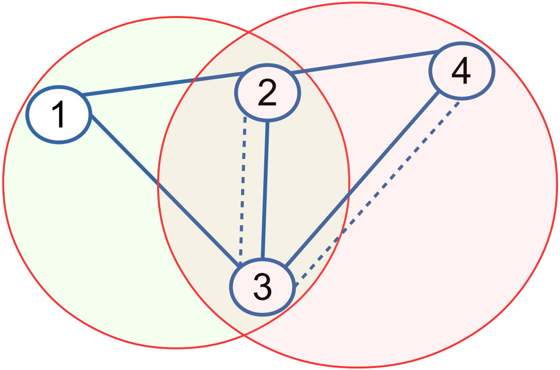


Figure: Two Triangular Interf Sets with a common conflict edge
Leading to a larger Strictly Rate 1/2 Subset

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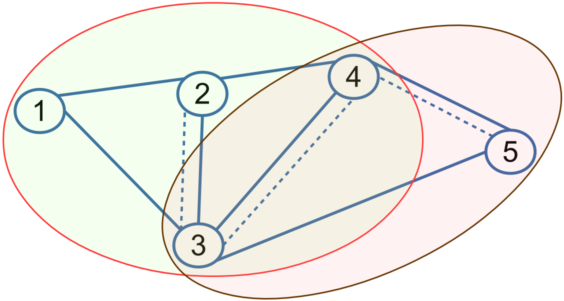

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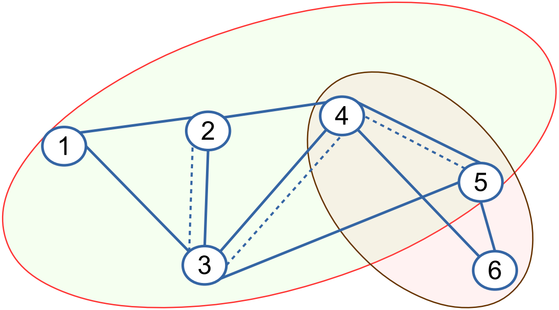


Figure: Maximal collection of 'Adjacent' Triangular Interfering sets: **Type-2 Set**

# Can go on to get larger strictly rate 1/2 sets...

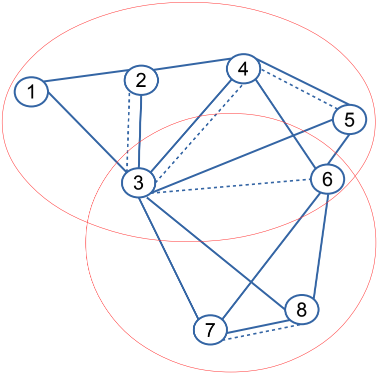


Figure: Maximal collection of Type-2 Sets: **Xtype-2 Set**

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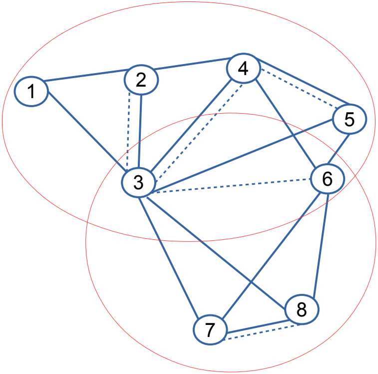


Figure: Maximal collection of Type-2 Sets: **Xtype-2 Set**

**Rate 1/3 necessary condition**: The Restricted-IC problem to a strictly rate 1/2 set must have a rate 1/2 solution. (poly time to check by [Jaf]).

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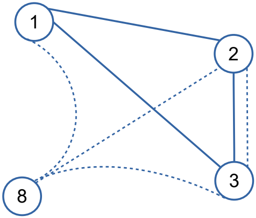
# Contractions of an IC problem



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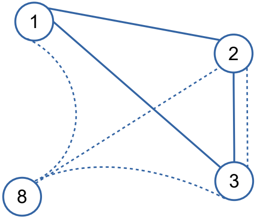


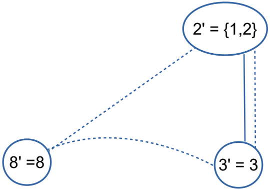


(d) Contracted in alignment edge (1,2)

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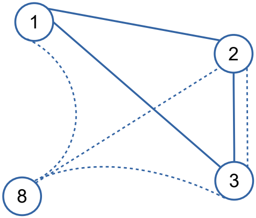


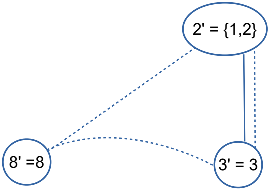


(f) Contracted in alignment edge (1,2)

**Maximal Contraction:** Successive contractions until no more is possible (every alignment edge has an associated conflict edge).

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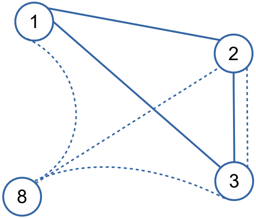
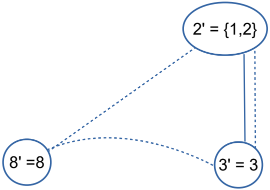



(h) Contracted in alignment edge (1,2)

**Maximal Contraction:** Successive contractions until no more is possible (every alignment edge has an associated conflict edge).

## Code for Contracted problem → Code for original problem

Let $\mathbb{I}'$ be a contraction of $\mathbb{I}$. Any scalar linear index coding solution of rate $R$ for $\mathbb{I}'$ can extended to a scalar linear index coding solution of rate $R$ for $\mathbb{I}$.

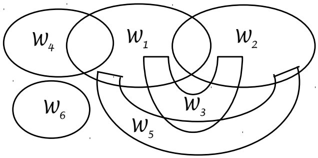
# A general feasible configuration - Sufficiency result

An IC problem $\mathbb{I}$ is rate $\frac{1}{3}$ feasible if there exists a maximal contraction $\mathbb{I}'$ of $\mathbb{I}$ such that the following conditions hold:

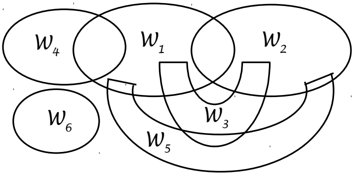
(a) IC problem restricted to any Xtype-2 set in $\mathbb{I}'$ must have a rate $\frac{1}{2}$ solution.

(b) No three Xtype-2 sets have a message vertex in common.

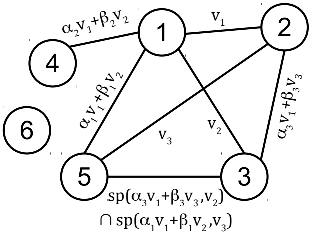
(c) For any two distinct Xtype-2 sets $\mathcal{W}_i, \mathcal{W}_j$, if $\mathcal{W}_i \cap \mathcal{W}_j \neq \phi$, then there is no conflict between any two messages in $\mathcal{W}_i \cap \mathcal{W}_j$.



# Proof Argument for Sufficiency




(i) Xtype-2 Sets




(j) Intersection Graph of Xtype-2 sets

- Random assignment of vectors from $\mathbb{F}^3$, but with constraints.
- Works with high prob, for large $|\mathbb{F}|$.

# Conclusions

- Rate 1/3 checking - NP Hard for fixed field size (Peeters, 1996 ). What about $|\mathbb{F}| \rightarrow \infty$?
- More forbidden structures, feasible structures.
- Other rational rates.

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R. Peeters, 'Orthogonal Representations over Finite Fields and the Chromatic Number of Graphs', 1996, Combinatorica.