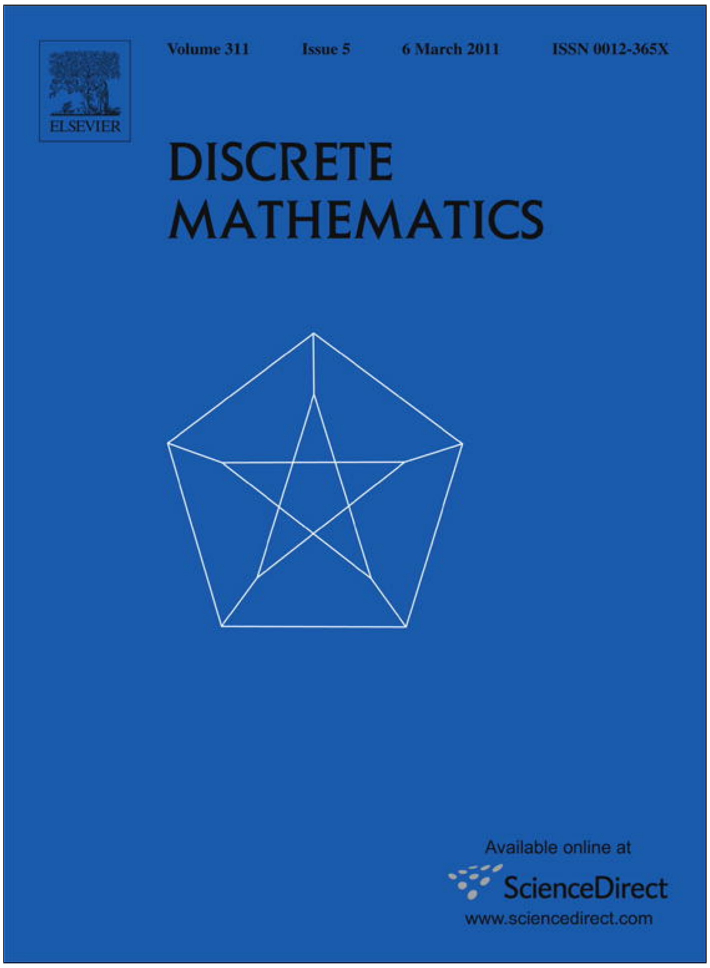
Volume 311 Issue 5 6 March 2011 ISSN 0012-365X
ELSEVIER
DISCRETE MATHEMATICS
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Discrete Mathematics 311 (2011) 342–348

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Acyclic vertex coloring of graphs of maximum degree 5☆

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ARTICLE INFO

Article history:
Received 31 March 2009
Received in revised form 25 October 2010
Accepted 31 October 2010
Available online 20 November 2010

Keywords:
Graph coloring
Acyclic coloring
Bounded degree graphs
Algorithms

ABSTRACT

An acyclic vertex coloring of a graph is a proper vertex coloring such that there are no bichromatic cycles. The acyclic chromatic number of G, denoted $a(G)$, is the minimum number of colors required for acyclic vertex coloring of graph G. For a family $\mathcal{F}$ of graphs, the acyclic chromatic number of $\mathcal{F}$, denoted by $a(\mathcal{F})$, is defined as the maximum $a(G)$ over all the graphs $G \in \mathcal{F}$. In this paper we show that $a(\mathcal{F}) = 8$ where $\mathcal{F}$ is the family of graphs of maximum degree 5 and give a linear time algorithm to achieve this bound.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A proper coloring of the vertices of a graph G is an assignment of colors to the vertices so that no two neighbors get the same color. A proper coloring is said to be *acyclic* if the coloring does not induce any bichromatic cycles. The acyclic chromatic number of a graph G is denoted by $a(G)$, and is the minimum number of colors required to acyclically color the vertices of G.

The concept of acyclic coloring of a graph was introduced by Grünbaum [14] and was further studied in the past two decades in several works, [3,1,6–8,4] among others. Determining $a(G)$ is a hard problem and Kostochka [15] proved that it is an NP-complete problem to decide for a given arbitrary graph G whether $a(G) \leq 3$.

Given the computational difficulty involved in determining $a(G)$, acyclic colorings have been studied for different families of graphs. In this context, Borodin [6] focuses on the family of planar graphs, Borodin et al. [9,8] focus on the family of planar graphs with "large" girth, and 1-planar graphs, respectively. Sopena [17] considers the family of outer planar graphs, and Fertin et al. [11] consider d-dimensional grids. The family of graphs of maximum degree 3 was considered by Grünbaum [14] and Skulrattanakulchai [16], and the family of graphs of maximum degree 4 was considered by [10].

Focusing on the family of graphs with a small fixed maximum degree, it was proved by Skulrattanakulchai [16] that $a(G) \leq 4$ for any graph of maximum degree 3. Burštein [10] showed that $a(G) \leq 5$ for any graph of maximum degree 4.

The work of Skulrattanakulchai was extended by Fertin and Raspaud [12] to show that it is possible to acyclically vertex color a graph G of maximum degree $\Delta(G)$ using at most $\Delta(G)(\Delta(G) - 1)/2$ colors. The same authors also showed that for any graph G of maximum degree 5, $a(G) \leq 9$, and there exists a linear time algorithm to acyclically color such a graph using at most 9 colors [12,13].

☆ This work is supported by the Department of Science and Technology, Govt. of India, New Delhi, under Project No. SR/S4/MS:406/06, and also DST-CMS project Lr.No. SR/S4/MS:516/07 Dt. 21-04-2008.

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doi:10.1016/j.disc.2010.10.024

Another direction that has yielded fruits is that of using the probabilistic method and the Lovász Local Lemma (LLL) [5]. Using this method, it was shown by Alon et al. [3] that any graph of maximum degree $\Delta(G)$ can be acyclically colored using $O(\Delta(G)^{4/3})$ colors. In the same paper, it was also shown that, as $n \to \infty$, there exist graphs with maximum degree $\Delta(G)$ that require $\Omega(\Delta(G)^{4/3}/(\log \Delta(G))^{1/3})$ colors for an acyclic coloring.

Alon et al. [3] further presented a polynomial time greedy algorithm to acyclically color any graph G with maximum degree $\Delta(G)$ using $\Delta(G)^2+1$ colors. This was later improved by Albertson et al. [2] to show that $a(G) \leq \Delta(G)(\Delta(G)-1)+2$.

In this paper, we improve the result of [12] for $\Delta(G) \leq 5$ and show that it is possible to acyclically color the vertices of a graph G with maximum degree at most 5 using only 8 colors. Below, we first introduce the notation that is used in the rest of the paper.

1.1. Notation

Let S refer to the set of positive integers $\{1, 2, \ldots, 8\}$. We stick to standard graph-theoretic notation (see [18]) for terms not defined here. We use notation from [16], and repeat it for the sake of completeness. We start with the following definition.

Definition 1.1. Let $W \subseteq V(G)$. The *neighborhood* of W, denoted $N(W)$, is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in W. A *neighbor* of W is a vertex in $N(W)$. We write $N(v)$ for $N(\{v\})$.

Definition 1.2. A *partial acyclic coloring* is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic coloring.

Suppose G has a partial acyclic coloring, and let α, β be any two colors. An *alternating* α, β*-path* is a path in G with alternate vertices colored α and β. An alternating path is a path that alternates some two colors. A path is odd (resp. even) if it contains an odd (even) number of edges. Let v be an uncolored vertex. We say that a color α is *available* for v if no neighbor of v is colored α. Similarly, we say that a color α is *feasible* for v if assigning the color α to v results in a partial acyclic coloring. (Thus feasibility implies availability but not the other way round.) Let C_v be a cycle in G containing vertex v. The cycle C_v is called α, β-dangerous if $C_v - v$ is an α, β-alternating path. A cycle C_v is *dangerous* if it is α, β-dangerous for some colors α, β. If there is more than one α, β-dangerous cycle through v for fixed α, β we consider them as the same types of dangerous cycles.

Definition 1.3. A vertex v is a *free vertex* if all its colored neighbors have distinct colors.

Notice that we consider a vertex v as a free vertex as long as its colored neighbors receive distinct colors. In essence, v can have uncolored neighbors and may still qualify as a free vertex. The notion of a free vertex is useful because recoloring is easy at free vertices.

1.2. Our results

In this paper, we show that any graph G with maximum degree at most 5 can be acyclically colored using 8 colors. We show this result by extending a partial acyclic coloring by one vertex at a time. Let v be the vertex that we color during an iteration. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex v, the vertex we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of v in all subcases. Specifically, we show the following lemmata which result in Theorem 1.7.

Lemma 1.4. *Let π be any partial acyclic coloring of G using colors in S, and let v be any uncolored vertex. If v has strictly less than 4 colored neighbors, then there exists a color α feasible for v. Moreover, such a color $\alpha \in S$ can be found in $O(1)$ time.*

Lemma 1.5. *Let π be any partial acyclic coloring of G using colors in S, and let v be any uncolored vertex. If v has exactly four colored neighbors, then there exists a partial acyclic coloring π_1 of G using colors in S and a color $\alpha \in S$ satisfying the following conditions.*

- *π_1 has the same domain as π.*
- *$\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$.*
- *α is feasible for v under π_1.*

Moreover, both π_1 and α can be found in $O(1)$ time.

Lemma 1.6. *Let π be any partial acyclic coloring of G using colors in S, and let v be any uncolored vertex. If v has exactly five colored neighbors, then there exists a partial acyclic coloring π_1 of G using colors in S and a color $\alpha \in S$ satisfying the following conditions.*

- *π_1 has the same domain as π.*
- *$\pi(x) \neq \pi_1(x)$ implies $x \in N(v)$ or $x \in N(N(v))$.*
- *α is feasible for v under π_1.*

Moreover, both π_1 and α can be found in $O(1)$ time.

Theorem 1.7. *The vertices of any graph G of degree at most 5 can be acyclically colored using eight colors in $O(n)$ time, where n is the number of vertices.*

To prove the above lemmata, we the use the following observations. First, we may assume that every vertex of G has exactly five neighbors. We consider v with neighbors u, w, x, y, and z. Suppose that the neighbors of v are colored with colors $\{c_1, \ldots, c_\ell\}$ and let $N_v(c_i)$ denote the number of neighbors of v that are colored with the color c_i. The following observation obtains the number of color pairs that can participate in dangerous cycles involving the vertex v. A similar observation is also used in [13].

Observation 1.8. *Let $v \in V(G)$ and the neighbors of v be u, w, x, y, and z which have colors in $\{c_1, \ldots, c_\ell\}$. Then, the number of color pairs that can participate in dangerous cycles involving the vertex v and two of its neighbors is at most $\sum_{c_i:N_v(c_i)\geq 2} 2N_v(c_i)$. The above bound generalizes to $\sum_{c_i:N_v(c_i)\geq 2} \left\lfloor \frac{N_v(c_i)(\Delta(G)-1)}{2} \right\rfloor$ for any graph of maximum degree $\Delta(G)$.*

The rest of the paper is organized as follows. In Section 2, we prove Lemma 1.4. In Sections 3 and 4, we prove Lemmata 1.5 and 1.6, respectively. Our algorithm uses techniques presented by Skulrattanakulchai [16], but our arguments are more involved. However, it is to be seen if there exist graphs of maximum degree 5 that still require 8 colors to acyclically color them.

2. Proof of Lemma 1.4

We split the proof of Lemma 1.4 into three subcases depending on the number of colored neighbors of v.

- *At most one neighbor of v is colored*: Now v has seven available colors and all of them are feasible for v.
- *v has exactly two colored neighbors*: Let them be w, x. Suppose $\pi(w) \neq \pi(x)$. As no neighbors of vertex v have the same color, the remaining six available colors are feasible for the vertex v. Suppose $\pi(w) = \pi(x)$. Since two colors are the same, the remaining seven colors are available to v. Out of the seven available colors, using Observation 1.8, at most four colors may be involved in four dangerous cycles. Hence, the other three colors are feasible for v.
- *v has exactly three colored neighbors*: We further subdivide this case into three subcases.
  - *No two neighbors of v have the same color*: In this case, v has five available colors and all of them are feasible for v.
  - *Two neighbors of v have the same color but the third neighbor has a different color from these two*: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, and $\pi(y) = 2$. Thus colors 3, 4, 5, 6, 7, and 8 are available for v. Any dangerous cycle involving v is $1, \beta$-dangerous for some β, and it involves the path $\{w, v, x\}$. As $\Delta(G) = 5$, no more than four colors can be the color β in any $1, \beta$-dangerous cycle involving v (see Observation 1.8). Thus, there are at least two colors feasible for v.
  - *All three colored neighbors of v have the same color*: Thus seven colors are available to v. Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$. Each colored neighbor of v that is, any vertex in $\{w, x, y\}$, can participate in at most four dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence, there are at most six dangerous C_v cycles. Since v has seven colors available but at most six of them are not feasible, at least one of them is feasible.

Thus, in all the above subcases, a feasible color for v can be found in $O(1)$ time.

3. Proof of Lemma 1.5

Without loss of generality, we suppose that all five neighbors of v exist. In this section we deal with the case where v has exactly four colored neighbors. We deal with the proof in several subcases depending on the colors of the neighbors of v.

- Case 1—*No colored neighbor of v has the same color*: Now v has four available colors, and all of them are feasible for v.
- Case 2—*Exactly two colored neighbors have the same color, and the remaining two colored neighbors have different colors*: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = 2$, and $\pi(z) = 3$. Any dangerous cycle involving v is $1, \beta$-dangerous for some β, and it involves the path $\{w, v, x\}$. As $\Delta(G) \leq 5$, no more than four colors can be the color β in any $1, \beta$-dangerous cycle involving v (see Observation 1.8). Since v has five available colors and at most four colors are not feasible, at least one of them is feasible.
- Case 3—*Exactly three colored neighbors have the same color, and the other colored neighbor is colored differently*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, and $\pi(z) = 2$. Now v has six available colors. Any dangerous cycle involving v is $1, \beta$-dangerous for some β and it involves the path $\{w, v, x\}$ or $\{x, v, y\}$ or $\{w, v, y\}$. Any vertex in $\{w, x, y\}$ can participate in at most four dangerous C_v cycles, and each C_v cycle contains two neighbors of v. (See Observation 1.8.) Assume that they are 1, 3-, 1, 4-,1, 5-, 1, 6-, 1, 7-, and 1, 8-dangerous, for otherwise there is an available color that is feasible to v. Assume without loss of generality that edge vw is involved in 1,3-, 1, 4-, 1, 5-, and 1, 6-dangerous cycles. This implies that the neighbors of w have colors 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 2 above.

- Case 4—*All four colored neighbors have the same color*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$. Any vertex in $\{w, x, y, z\}$ can participate in at most four dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence there are at most eight dangerous C_v cycles (see Observation 1.8). Thus none of the available colors may be feasible for v. We now deal with two subcases.
  - *At least one of* $\{w, x, y, z\}$ *is a free vertex*: Assume without loss of generality that w is a free vertex and neighbors of w have colors 2, 3, 4, and 5. Define π_1 by setting $\pi_1(w) = 6$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1, this case reduces to Case 3 above.
  - *None of* $\{w, x, y, z\}$ *is a free vertex*: In this case, any vertex in $\{w, x, y, z\}$ has at most three differently colored neighbors. Thus, any neighbor of v can participate in at most three dangerous C_v cycles, and each C_v cycle contains two neighbors of v; hence, there are at most six types of dangerous C_v cycles. Since v has seven available colors but at most six of them are not feasible, at least one of them is feasible.
- Case 5—*Exactly two neighbors have the same color, and another two neighbors have the same (but different from the above) color*: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, and $\pi(y) = \pi(z) = 2$. There are at most four $1, \beta$-dangerous C_v cycles for some β involving the path $\{w, v, x\}$ and at most four $2, \beta$-dangerous C_v cycles for some β involving the path $\{y, v, z\}$ (see Observation 1.8). Hence, presently, no color is feasible for v. To find a feasible color for v, we make the following case distinction.
  - *At least one of the neighbors of* v *is a free vertex*: Assume without loss of generality that w has neighbors with colors 3, 4, 5, 6. Define π_1 by setting $\pi_1(w) = 7$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1, this case reduces to Case 2 of this section.
  - *None of* $\{w, x, y, z\}$ *is a free vertex*: If no available color is feasible for v, then there must be three types of $1, \beta$-dangerous C_v cycles and three types of $2, \beta$-dangerous C_v cycles. Assume without loss of generality that there are 1, 3-, 1, 4-, 1, 5-, and 2, 6-, 2, 7-, 2, 8-dangerous cycles. This implies that the three neighbors of w are colored 3, 4, 5, and another neighbor should be colored with any of $\{3, 4, 5\}$. Let the like colored neighbors of w be w_1, w_2, and let w_1 and w_2 be colored with the color 3. The available colors for recoloring w are any of $\{2, 6, 7, 8\}$.

    If $N(w_1)$ and $N(w_2)$ both contain neighbors colored with colors $\{2, 6, 7, 8\}$, then there is no 1,3-dangerous cycle involving v; hence, we can color v with color 3. Otherwise, one of the colors in $\{2, 6, 7, 8\}$, say k, is missing in the neighborhood of at least w_1 or w_2. In this case, we define a partial acyclic coloring π_1 by setting $\pi_1(w) = k$ for $k \in \{2, 6, 7, 8\}$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Then this case reduces to Case 3 if $k = 2$ and reduces to Case 2 if $k \in \{6, 7, 8\}$.

Thus, in this case also we can find a feasible color for v in $O(1)$ time. Note that, we had to recolor vertices.

4. Proof of Lemma 1.6

Suppose v has five colored neighbors. We deal with the proof of Lemma 1.6 by breaking it into several subcases.

- Case 1—*No neighbor of* v *have the same color*: Now v has three available colors, and all of them are feasible for v.
- Case 2—*Two neighbors have the same color, and three are differently colored*: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = 2$, $\pi(z) = 3$, and $\pi(u) = 4$. Any dangerous cycle involving v is $1, \beta$-dangerous for some β, and it involves the path $w - v - x$. Since $\Delta(G) = 5$, no more than four colors can be the color β in any $1, \beta$-dangerous cycle involving v (see Observation 1.8). Assume that they are 1, 5-, 1, 6-, 1, 7-, and 1, 8-dangerous cycles. Otherwise there is an available color that is feasible to v. This implies that neighbors of w are colored 5, 6, 7, and 8. Define π_1 by setting $\pi_1(w) = 2$, $\pi_1(x) = 2$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1, the color 1 is feasible for v. Thus, finding the asserted π_1 can be done in $O(1)$ time.
- Case 3—*Three neighbors have the same color, and two neighbors are differently colored*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, $\pi(z) = 2$, and $\pi(u) = 3$. Any of $\{w, x, y\}$ can participate in at most four dangerous cycles, and each C_v cycle contains two neighbors of v; hence there are at most six dangerous C_v cycles (see Observation 1.8). Thus, none of the five available colors may be feasible for v. So, we distinguish between two subcases.
  - *At least one of* $\{w, x, y\}$ *is a free vertex*: Assume without loss of generality that w has neighbors colored 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 2 of Section 4.
  - *None of* $\{w, x, y\}$ *is a free vertex*: Any vertex in $\{w, x, y\}$ has at most three differently colored neighbors. Thus, any neighbor of v can participate in at most three dangerous C_v cycles, and each C_v cycle contains two neighbors of v; hence there are at most four types of dangerous C_v cycles. Since v has five available colors but at most four of them are not feasible, at least one of them is feasible. Such a feasible color can be found in $O(1)$ time.
- Case 4—*Two neighbors have the same color, and two others have the same (but different from the above) color, and another one is colored differently*: Assume without loss of generality that $\pi(w) = \pi(x) = 1$, $\pi(y) = \pi(z) = 2$, and $\pi(u) = 3$. There are at most four $1, \beta$-dangerous C_v cycles for some β involving the path $w - v - x$ and at most four $2, \beta$-dangerous C_v cycles for some β and involves the path $y - v - z$ (see Observation 1.8). Since v has five available colors and all of them can be involved in dangerous C_v cycles, no color may be feasible for v. So we proceed by making the following case distinction.

– *At least one of* $\{w, x, y, z\}$ *is a free vertex*: Assume without loss of generality that w has neighbors colored 4, 5, 6, and 7. Define π_1 by setting $\pi_1(w) = 8$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 2 of Section 4.
– *None of* $\{w, x, y, z\}$ *is a free vertex*: For no available color to be feasible for v there must be at least three types of $1, \beta$-dangerous C_v cycles and two types of $2, \beta$-dangerous C_v cycles, or vice versa. Assume without loss of generality that they are 1, 4-, 1, 5-, 1, 6-, and 2, 7-, 2, 8-dangerous C_v cycles. This implies that both w and x have three of their neighbors with colors in $\{4, 5, 6\}$ and the remaining neighbor should have a color among $\{4, 5, 7\}$. Let the like colored neighbors of w be w_1, w_2, and let w_1 and w_2 have color 4. The available colors for recoloring w are $\{2, 3, 7, 8\}$. When any of $\{2, 7, 8\}$ is missing in the neighborhood of any of w_1 or w_2, we define π_1 by setting $\pi_1(w) = k$, where k is one among the missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 3 of Section 4 if $k = 2$, or Case 2 of Section 4 if $k \in \{7, 8\}$. Otherwise, we distinguish between the following subcases.
  * *At least one of* w_1 *and* w_2 *is a free vertex*: Note that, w_1, w_2 have neighbors with colors $\{2, 7, 8\}$. Let the other neighbor of w_1 and w_2 be colored a and b, respectively. (If those nodes are not colored yet, we take the colors to be -1). We define π_1, a partial acyclic coloring, as follows:

$$\pi_1(w) = 7$$
$$\pi_1(w_1) = \begin{cases} 1, & a \neq 1 \\ 3, & a = 1 \end{cases}$$
$$\text{and,} \quad \pi_1(t) = \pi(t) \quad \text{for every other colored vertex } t.$$

  * *None of* w_1 *and* w_2 *is a free vertex*; In this case we make the following observations.

Observation 4.1. *There is no possibility of* $1, 4$-*dangerous cycles.*

Observation 4.2. *If there is no possibility of* $2, 4$-*dangerous cycles, then the color* 4 *is feasible for* v. *Otherwise all the neighbors of both* y *and* z *should have colors in* $\{4, 7, 8\}$ *only.*

To continue further, consider the vertex x, and let the like neighbors of x be x_1, and x_2. If any of $\{2, 7, 8\}$ is missing in the neighborhood of any of x_1 or x_2, then define π_1 by setting $\pi_1(x) = k$, where k is one among the missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 3 of Section 4 if $k = 2$, or Case 2 of Section 4 if $k \in \{7, 8\}$.
Otherwise if color 3 is missing in the neighborhood of any of x_1 or x_2, then define π_1 by setting $\pi_1(x) = 3$, $\pi_1(w) = 3$, and $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 color 1 is feasible for v as there is no possibility of 2, 1- and 3, 1-dangerous cycles under π_1 (using Observations 4.1 and 4.2). For an illustration, we refer the reader to Fig. 1.
Otherwise x_1, and x_2 have neighbors with colors $\{2, 3, 7, 8\}$. Then define π_1 by setting $\pi_1(x) = 7$, $\pi_1(x_1) = 1$, and $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 2 of Section 4.

- Case 5—*Four neighbors have the same color, and one neighbor is differently colored*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = 1$, and $\pi(u) = 2$. Any of $\{w, x, y, z\}$ can participate in at most four dangerous cycles, and each such C_v cycle contains two neighbors of v; hence there can be at most eight dangerous C_v cycles (see Observation 1.8). Thus none of the six available colors may be feasible for v. We proceed in the following way by making a case distinction.
  – *At least one of* $\{w, x, y, z\}$ *is a free vertex*: Assume without loss of generality that w has neighbors with colors 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 3 of Section 4.
  – *None of* $\{w, x, y, z\}$ *is a free vertex*: Any vertex in $\{w, x, y, z\}$ has at most three differently colored neighbors. Thus any neighbor of v can participate in at most three dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence there are at most six types of dangerous C_v cycles. Assume without loss of generality that they are 1, 3-, 1, 4-, 1, 5-, 1, 6-, 1, 7-, and 1, 8-dangerous C_v cycles. (Otherwise, there is obviously an available color that is feasible for v.) This implies that every available color is in the neighborhood of exactly two of the vertices in $\{w, x, y, z\}$. Assume without loss of generality that w has neighbors colored 3, 3, 4, and 5. Let the like neighbors of w be w_1 and w_2, and w_1 and w_2 have the color 3.
  If any of the colors in $\{6, 7, 8\}$ is missing in the neighborhood of any of w_1 or w_2, then define π_1 by setting $\pi_1(w) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 3 of Section 4. Otherwise, w_1 and w_2 have neighbors with colors in $\{6, 7, 8\}$. Let the other neighbor of w_1 and w_2 be colored a and b, respectively. (If those nodes are not colored, we take them to be colored with -1).
  Consider the neighbor of w_1 colored a. When $a \notin \{6, 7, 8\}$, define π_1 as follows.

$$\pi_1(w) = 6$$
$$\pi_1(w_1) = \begin{cases} 2 & \text{when } a \neq 2, \\ 1 & \text{when } a = 2, \end{cases} \quad \text{and}$$
$$\pi_1(t) = \pi(t), \quad \text{for every other colored vertex } t.$$

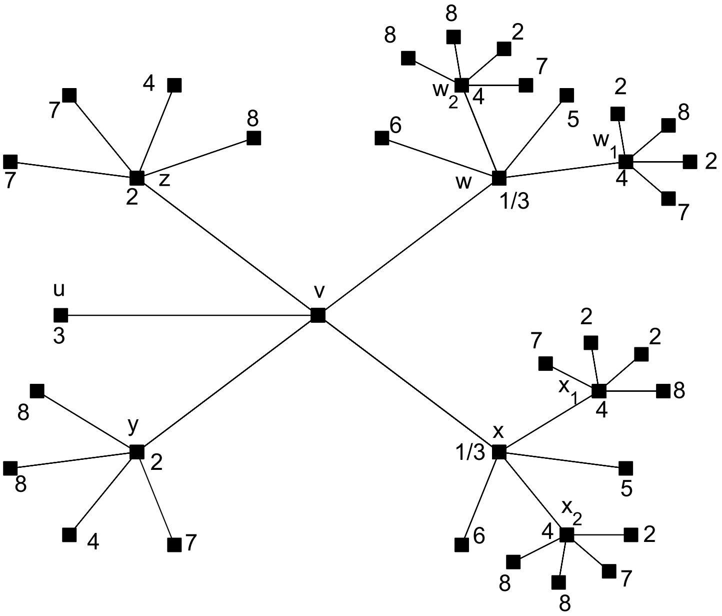

Fig. 1. The figure applicable to Case 4 of the proof of Lemma 1.6 illustrates the recoloring of vertices w and x to color 3, the colors a and b (see Case 4 in the proof of Lemma 1.6) are 2 and 7, respectively. The neighbors of z and y are colored only among colors 4, 7, and 8 as shown. In the figure a vertex v with label p/q means that the vertex v that is colored p is recolored q as part of our algorithm. The neighbors of u are not shown in this figure. After recoloring, color 1 is thus made feasible for v.

Moreover, under π_1 this reduces to Case 3 of Section 4. If $a \in \{6, 7, 8\}$, then consider the neighbor of w_2 colored b. If $b \notin \{6, 7, 8\}$, then define π_1 similarly as above. When $b \in \{6, 7, 8\}$, then notice that there is no possibility of 1, 3-dangerous cycles through v involving w. As we stated earlier every available color is in the neighborhood of exactly two of the vertices in $\{w, x, y, z\}$. Apart from w there is only one vertex among x, y, z that has a neighbor colored 3. Hence there is no 1, 3-dangerous cycle through v. Thus, color 3 is feasible for v.

- Case 6—*Three neighbors have the same color, and the two remaining neighbors have the same (but different from above) color*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = 1$, and $\pi(z) = \pi(u) = 2$. Any of $\{w, x, y\}$ can participate in at most four 1, β-dangerous cycles, and each such C_v cycle contains two neighbors of v; hence, there are at most six 1, β-dangerous C_v cycles. Similarly, there are at most four 2, β-dangerous C_v cycles (see Observation 1.8). Hence, none of the available colors cannot be feasible for v. During the recoloring process, we distinguish between the following cases.
  - *At least one of* $\{w, x, y, z, u\}$ *is a free vertex*: Assume without loss of generality that w has neighbors colored 3, 4, 5, and 6. Define π_1 by setting $\pi_1(w) = 7$ and $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1, this case reduces to Case 4 of Section 4. If, on the other hand if any of z or u is a free vertex, then a similar recoloring reduces this to Case 3 of Section 4.
  - *None of* $\{w, x, y, z, u\}$ *is a free vertex*: In this case, any vertex in $\{w, x, y\}$ has at most 3 differently colored neighbors. Thus, any of $\{w, x, y\}$ can participate in at most three types of dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence, there are at most four 1, β-dangerous C_v cycles. By a similar reasoning, there can be at most three 2, β-dangerous cycles. we now distinguish between following cases.
    - *The neighborhood of at least one of* z *and* u *has exactly three colors*: Assume without loss of generality that these are 3, 4, and 5. Hence, z has three neighbors with colors in $\{3, 4, 5\}$ and the remaining neighbor has a color among $\{3, 4, 5\}$. Let it have color 3. Let the like neighbors of z be z_1 and z_2, that is, both z_1 and z_2 are colored with 3. If any of $\{1, 6, 7, 8\}$ is missing in the neighborhood of any of z_1 or z_2, then define π_1 by setting $\pi_1(z) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1, this case reduces to Case 3 of Section 4 if $k \in \{6, 7, 8\}$, or to Case 5 of Section 4, if $k = 1$.

      Otherwise, both z_1 and z_2 have neighbors with colors in $\{1, 6, 7, 8\}$. Then we recolor as follows. Define π_1 by setting $\pi_1(z) = 6$, $\pi_1(z_1) = 2$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Moreover, under π_1, this case reduces to Case 3 of Section 4.
    - *Both* z *and* u *have all neighbors with at most two colors*: In this case, for no color to be feasible for v, there must be four types of 1, β-dangerous cycles and two types of 2, β-dangerous cycles. Assume without loss of generality that there are 1, 3-, 1, 4-, 1, 5-, 1, 6-, and 2, 7-, 2, 8-dangerous C_v cycles. For this to happen, the neighborhood of z and u is with colors 7 and 8 only. Now let us consider the neighborhood of vertices w, x, and y. If all of $\{w, x, y\}$ have only two differently colored neighbors, then every one of $\{w, x, y\}$ can participate in at most two dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence there can be at most three types of 1, β-dangerous C_v cycles. Thus, at most three available colors can participate in dangerous cycles and the other available color cannot participate in dangerous cycles. Thus, it must be the case that two neighbors exist with three differently colored

neighbors so that no available color is feasible for v. Let the two such neighbors be w and x. Assume without loss of generality that w has neighbors with colors 3, 3, 4, and 5. Let the like neighbors of w be w_1 and w_2, that is, both w_1 and w_2 have color 3.

If any of the colors $\{6, 7, 8\}$ is missing in the neighborhood of any of w_1 or w_2, then define π_1 by setting $\pi_1(w) = k$, where k is one of the missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 4 of Section 4. Otherwise, we distinguish between the following cases.

(a) *At least one of w_1 and w_2 is a free vertex*: Note that, w_1 and w_2 have neighbors colored with colors $\{6, 7, 8\}$. Let the other neighbor of w_1 and w_2 be colored a and b, respectively. (If those nodes are not colored yet, we take the colors to be -1). We define π_1, partial acyclic coloring as follows:

$$\pi_1(w) = 6$$
$$\pi_1(w_1) = \begin{cases} 2 & \text{when } a \neq 2, \\ 1 & \text{when } a = 2, \end{cases} \quad \text{and}$$
$$\pi_1(t) = \pi(t), \quad \text{for every other colored vertex } t.$$

Moreover, under π_1 this reduces to Case 4 of Section 4.

(b) *None of w_1 and w_2 is a free vertex*: Now define π_1 by setting $\pi_1(w) = 2$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Then, under π_1, w cannot participate in any dangerous cycles and the four $1, \beta$-dangerous cycles should be formed through $x - v - y$ path only. This implies that x and y are free vertices, which is not true according to our earlier assumption. Hence, under π_1 there is an available color that is feasible for v, which can be found by looking at the colored neighbors of x and y.

- Case 7—*All five neighbors have the same color*: Assume without loss of generality that $\pi(w) = \pi(x) = \pi(y) = \pi(z) = \pi(u) = 1$. Any of $\{w, x, y, z, u\}$ can participate in at most four dangerous C_v cycles, and each C_v cycle contains two neighbors of v; hence there can be at most ten dangerous C_v cycles (see Observation 1.8). Thus, none of the seven available colors may be feasible for v; hence, we distinguish between the following cases.
  - *At least one of $\{w, x, y, z, u\}$ is a free vertex*: Assume without loss of generality that w has neighbors colored 2, 3, 4, and 5. Define π_1 by setting $\pi_1(w) = 6$, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 5 of Section 4.
  - *None of $\{w, x, y, z, u\}$ is a free vertex*: Any vertex in $\{w, x, y, z, u\}$ has at most three differently colored neighbors. Assume that every neighbor of v has two differently colored neighbors. Then every neighbor of v can participate in at most two types of dangerous C_v cycles, and each such C_v cycle contains two neighbors of v; hence there are at most five types of dangerous C_v cycles. Thus, at most five available colors can participate in dangerous cycles and the other two available colors cannot participate in dangerous cycles. So we can find a feasible color for v.

    Otherwise, there exist two neighbors with three differently colored neighbors so that no available color is feasible for v. Let the two neighbors be w and x. Assume without loss of generality that w has neighbors colored 2, 2, 3, and 4. Let the like neighbors of w be w_1 and w_2, and w_1 and w_2 have color 2. If any of $\{5, 6, 7, 8\}$ is missing in the neighborhood of any of w_1 or w_2, then define π_1 by setting $\pi_1(w) = k$, where k is one among missing colors, and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 5 of Section 4. Otherwise, w_1 and w_2 have neighbors colored $\{5, 6, 7, 8\}$. Then, define π_1 by setting $\pi_1(w) = 5$, $\pi_1(w_1) = 1$ and setting $\pi_1(t) = \pi(t)$ for every other colored vertex t. Note that, π_1 is also a partial acyclic coloring. Moreover, under π_1 this case reduces to Case 5 of Section 4.

Thus, in this case also a feasible color for v can be found in $O(1)$ time. However, we needed to recolor some vertices in the 2-neighborhood of v in some subcases.

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